Jordan Types of Artinian Algebras and Geometry of Punctual Hilbert Schemes

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Titles and Abstracts

A refinement of the local cactus algorithm
Alessandra Bernardi, Università di Trento

We present a refined version of the cactus algorithm in the case of local schemes, improving both reliability and efficiency in describing the local structure of Artinian Gorenstein schemes.

Jordan type stratification of spaces of commuting nilpotent matrices
Mats Boij, KTH

The Jordan type of a nilpotent matrix in the dense orbit of the nilpotent commutator of a given nilpotent matrix of Jordan type P is *stable*, which means that the parts differ by at least two. Fixing a matrix J of stable Jordan type Q, there is an affine space of nilpotent matrices commuting with J.

In recent joint work with A. Iarrobino and L. Khatami, we use some tropical calculations to determine equations defining the loci of each partition P for which Q is the generic commuting partition. We also propose a conjecture generalizing this result to arbitrary stable Q. A key ingredient is the recent proof of the Box Conjecture by J. Irving, T. Košir and M. Mastnak.

Hilbert functions of Veronesean subvarieties and Complete Intersections
Stefano Canino, Universytet Warszawski

If X is a set of reduced points lying on a rational normal curve, the Hilbert function of X is classically known. Starting from this result, we address the following problem: what are the possible Hilbert functions of a reduced subvariety of a Veronese variety? We provide a general result for any Veronese variety and then derive an effective characterisation of the Hilbert function of points lying on a Veronese surface. As an application, we completely classify the complete intersections of the ambient space that lie on a Veronese surface, and we formulate a conjecture for Veronese varieties in higher dimensions. This is joint work with Prof. Enrico Carlini.

K-stability of nets of conics
Ivan Cheltsov, University of Edinburgh

Nancy Abdallah, Jacques Emsalem and Tony Iarrobino classified net of conics in the paper "Nets of Conics and associated Artinian algebras of length 7", which is an extended version of the

unpublished preprint "Réseaux de coniques et algèbres de longueur sept associées" by Jacques and Tony written back in 1972. In my talk, I will explain how to use this results to determine which smooth Fano 3-folds in the deformation family No. 2.24 (divisors of degree $(1,2)$ in the product of two projective planes, see `https://www.fanography.info/`) admit Kahler-Einstein metric and which does not.

Low degree equations for the Hilbert schemes

Laurent Evain, Université d'Angers

What are the equations for the Hilbert schemes in their natural embeddings? Different set of (high degree) equations have been found. Among them, the determinantal equations by Tony Iarrobino and Steve Kleiman.

In this talk, a new construction of the Hilbert scheme will be given, from which we derive a set of equations of degree one and two.

Our construction extends the description by Nakajima of $\mathrm{Hilb}^p(\mathbb{A}^2)$ to the projective case and a non constant polynomial. The Hilbert scheme is described as a quotient of a scheme of quiver representations.

The equations are reminiscent of the Plücker relations for Grassmannians: they are explicit and built formally with permutations on indexes on the Plücker coordinates.

Irreducible components of Hilbert schemes and application to secant varieties

Maciej Gałązka, Daegu-Gyeongbuk Institute of Science and Technology

We recall the connection of the Gorenstein loci in the Hilbert scheme of points on a projective variety X with the cactus varieties of X in some embedding. We give an analogous description of cactus varieties of pencils (which correspond to simultaneous rank). We conclude by showing some examples of cactus varieties of pencils which are equal to their corresponding secant varieties. This is a complementary talk to Klemen Šivic's presentation, and is based on joint work with Hanieh Keneshlou and Klemen Šivic.

Dynamics of eigenvalues of random matrices under iterated transforms.

André Galligo, Université Côte d'Azur

The point process limit of the random matrix spectra are defined via the limit joint densities, relying on some algebraic structures. I will describe some relations between these dynamics and the dynamics of root sets of some random polynomials under iterated application of differential operations on the polynomials.

Cox-Gorenstein Algebras

Rodrigo Gondim, Universidade Federal Rural de Pernambuco

We study G-graded Artinian algebras having Poincaré duality and their Lefschetz properties. We prove the equivalence between the toric setup and the G-graded one. We prove a Hessian criterion in the G-graded setup. We provide an application to toric geometry. We propose a question about Jordan types.

(Joint with U. Bruzzo, R. Holanda, W. Montoya.)

Around logarithmic forms and vector fields, with a view on their modules of residues
Michel Granger, Université d'Angers

In this talk we recall Saito's theory about the notions mentioned in the title for an hypersurface. We shall focus more specifically the notion of residue and prove duality statements for the module that they form. Concerning applications we gave the final step of a proof of Saito's conjecture about the characterisation of singularities which are normal crossings in codimension 2. We mention also a characterisation of singularities with Gorenstein singular locus. Depending on time left we shall develop more examples of residue modules, for curves, hyperplane arrangement and/or sketch a generalisation of the theory to singularities in higher codimension.

Solving Linear Equations
I-Chiau Huang, Institute of Mathematics, Academia Sinica

We introduce the notion of an "initial condition" for a module M over a commutative Noetherian local ring $(A, \mathfrak{m})$, allowing for a recursive construction of its "solution modules". If M has zero-dimensional support, such as the residue field of A, we demonstrate that the solution module $E(M)$ is its "linear closure", turning out to be an injective hull of M. The construction of $E(M)$ for finitely generated M, hence injective hull of M, is explicit and computable, devoid of the need for Zorn's lemma. As an application, we improve Baer's criterion for a module N with zero-dimensional support to be injective: If any A-homomorphism from $\mathfrak{m}$ to N lifts to A, then N is injective. For the case that A is Artinian and M is its residue field, we demonstrate how systems of linear equations are explicitly solved to obtain an injective hull.

Iarrobino's symmetric decomposition: open questions and some developments
Joachim Jelisiejew, Uniwersytet Warszawski

We will discuss how Iarrobino's symmetric decomposition became a central numeric invariant for Gorenstein algebras and discuss the current state of the art: open questions and results related to completed quadrics and Iarrobino's scheme.

Maximal Hilbert functions of Artinian quotients of a product ring.
Jan Kleppe, Oslo Metropolitan University

Given a field k and a graded k-algebra A, let $\mathbb{F}\Psi_A^{\mathbf{h}}$ and $\mathbb{H}\Psi_A^{\mathbf{h}}$ be the schemes parameterizing filtered quotients and graded quotients of A with Hilbert function $\mathbf{h}$. Let $\mathbb{F}\Psi_A^{\mathbf{h},\mathbf{t}}$ and $\mathbb{H}\Psi_A^{\mathbf{h},\mathbf{t}}$ be their subschemes of Artinian quotients of socle type $\mathbf{t}$.

In 1984, Iarrobino proved that, if k is infinite, if A is a polynomial ring, if $\mathbf{t}$ is permissible in a certain sense, and if $\mathbf{h} = \mathbf{h}^I$ where

$$\mathbf{h}^I(p) := \min\{a(p), sum_{q>0} t(q)a(q-p)\}$$

and $a(i) := dim A_i$, then $\mathbb{F}\Psi_A^{\mathbf{h},\mathbf{t}}$ is an affine space bundle over $\mathbb{H}\Psi_A^{\mathbf{h},\mathbf{t}}$, and $\mathbb{H}\Psi_A^{\mathbf{h},\mathbf{t}}$ is nonempty, irreducible and covered by open subschemes, each isomorphic to $\mathbb{A}^N$ with N explicit. For any A, there's a similar maximal $\mathbf{h}$, but it's not necessarily equal to $\mathbf{h}^I$.

In this talk, we analyze the case where $A := S \times_k T$ and $\mathbf{h} \neq \mathbf{h}^I$. When $S := k[x]$, a polynomial ring in one variable, we prove that $\mathbb{F}\Psi_A^{\mathbf{h},\mathbf{t}}$ and $\mathbb{H}\Psi_A^{\mathbf{h},\mathbf{t}}$ are close to be as nice as when $\mathbf{h} = \mathbf{h}^I$. In 2001, Cho and Iarrobino gave such examples with $T := k[y,z]/(z^5)$ in the graded case. The new work described here is joint work with Steve Kleiman.

A proof of the Box Conjecture for commuting pairs of matrices
Tomaž Košir, Univerzi v Ljubljani

We will review some of the work on commuting pairs of matrices that led to the Box Conjecture of Anthony Iarrobino and his collaborators. Then, we will sketch a proof of the Conjecture. The proof hinges naturally on the Burge correspondence between the set of all partitions and a set of binary words. For connection with the algebraic and geometric setup of matrices and nilpotent orbits we use Shayman's results on invariant subspaces of a nilpotent matrix. This is joint work with John Irving and Mitja Mastnak.

Families of symmetric and skew-symmetric matrices and vector bundles
Emilia Mezzetti, Università degli Studi di Trieste

I will report on joint work with A. Boralevi and M.L. Fania, about linear spaces and quadric surfaces contained in the Pfaffian hypersurface in $\mathbf{P}^{14}$. I will then explain the connections with congruences of lines in the 5-dimensional projective space and with vector bundles.

The weak Lefschetz property for Artinian Gorenstein algebras
Rosa Maria Miró-Roig, Universitat de Barcelona

It is an extremely elusive problem to determine which standard artinian Gorenstein graded K-algebras satisfy the weak Lefschetz property (WLP). Codimension 2 artinian Gorenstein graded K-algebras have the WLP and it is open to what extent such result might work for codimension 3 artinian Gorenstein graded K-algebras.

In this talk I will summarize what we know about this problem and, in particular, I will show that all artinian Gorenstein K-algebras of codimension 3 with arbitrary socle degree d, arbitrary Sperner number α and at least three peaks (i.e., three values of the h-vector reaching the Sperner number) do satisfy the WLP.

Tensor decomposition via the analysis of Artinian algebras
Bernard Mourrain, Université Côte d'Azur

Tensor decomposition is a challenging problem from a computational and numerical point of view. This can be explained by the complex and still unrevealed geometry hidden behind the scene.

We will explore this complex geometry via the lens of algebra, investigating how Artinian algebras can be naturally associated to general (additive) decompositions of tensors. We will review algebraic geometric approaches, which reduces tensor decomposition to direct eigenvalue and eigenvector computations when the rank is small.

When the rank is higher, the geometry is much more complex. We will see how the tensor decomposition problem can be transformed into a simultaneous diagonalization problem of extended tensors or extensor. We will connect the varieties of extensors with the punctual Hilbert scheme, via families of commuting matrices and illustrate the concepts and the computational approach on some examples.

Results and open problems passing from a local ring to its associated graded ring
Maria Evelina Rossi, Università degli Studi di Genova

Let (A, m) be a complete local ring and $G = \mathrm{gr}_m(A)$ its associated graded ring. The problem of the descent of a property from G to A was extensively studied and the answers are predominantly positive. The problems arise passing from A to G, because we may lose many good geometric

and algebraic properties. We present an overview up to a recent work with A. De Stefani and M. Varbaro. We introduce a homogenization technique which allows to relate G to the special fiber and A to the generic fiber of a "Groebner-like" deformation. Using this technique we prove sharp results concerning the connectedness of A and G.

Lefschetz properties of the Artinian Orlik-Terao algebra
Hal Schenck, Auburn University

Some 30 years ago, Peter Orlik and Hiro Terao introduced a commutative analog of the Orlik-Solomon (OS) algebra. The OS algebra is the cohomology ring of a hyperplane arrangement complement, and is a quotient of an exterior algebra by a combinatorially determined ideal. The Orlik-Terao (OT) algebra and Artinian version (AOT) have subsequently been studied by many authors (sometimes under the guise as the "algebra of reciprocal forms"). It has surprising connections to classical algebraic geometry (for example, to certain blowups of projective space). We analyze Lefschetz properties of the Artinian Orlik-Terao algebra.

My Favorite Problems Inspired by Tony Iarrobino's Work
Alexandra Seceleanu, University of Nebraska-Lincoln

Tony Iarrobino's deep insights into the structure of Artinian Gorenstein algebras, Hilbert schemes, and the geometry of punctual schemes have left a lasting mark on modern algebraic geometry and commutative algebra. In this talk, I will share a collection of my favorite problems, most of which are still open, that are inspired by his work. These include questions about the properties of Artinian ring and the homological and asymptotic behavior of punctual schemes and related Macaulay inverse systems. Along the way, I'll highlight the powerful techniques developed or influenced by Tony and discuss how his vision continues to inspire new directions in the field.

Irreducible components of Hilbert schemes of points
Klemen Šivic, Univerzi v Ljubljani

In the talk we classify irreducible components of Hilbert schemes of 9 and 10 points in affine spaces of any dimension. The main tool is the connection between Hilbert schemes of points and varieties of commuting matrices. This is joint work with Maciej Gałązka and Hanieh Keneshlou.

Regularity of tangential decompositions
Daniele Taufer, KU Leuven

Given a form F of degree d, its tangential decompositions are additive decompositions of F that involve only terms of type $L^{d-1}G$, where L and G are linear forms. To any such decomposition, we can naturally associate 0-dimensional apolar schemes made of simple points (when $L = G$, projectively), and 2-jets (when L and G are not proportional). Among these schemes, it is possible to find irredundant ones (i.e. those minimal by inclusion) that are not regular in degree d. Nonetheless, this never happens for the shortest schemes (i.e. those minimal by length), as we can always "refine" such an irregular scheme to obtain a d-regular and strictly shorter one, of the same type. In this talk, I will illustrate this construction and briefly discuss its implications for the explicit computation of minimal tangential decompositions of forms. This is based on a joint work with A. Bernardi and A. Oneto.

On the definition of a general element and polarization

Junzo Watanabe, Tokai University

Define $D_{yx} = y_1 \frac{\partial}{\partial x_1} + \cdots + y_n \frac{\partial}{\partial x_n}$. This is called a polarization. In the classical invariant theory this is used to create a new invariant from a known invariant. In recent years this is used effectively in (1) and (2). In this talk I want to draw attention to the fact D_{yx}, D_{xy}, $[D_{yx}, D_{xy}]$ are an $\mathfrak{sl}_2$--triple acting on the homogeneous space of degree d of the graded algebra $K[y_1, \ldots, y_n][x_1, \ldots, x_n]$. It gives us new problems and new examples of Gorenstein algebras satisfying/failing SLP.

(1) P. Brändén and J. Huh, "Lorentzian polynomials," 2020.

(2) P. Macias Marques, M. McDaniel and A. Secelianu, "Higher Lorentzian polynomials, higher Hessians, and the Hodge Riemann property for graded oriented Artinian Gorenstein algebras in codimension two," 2024.