Tensor decomposition via the analysis of Artinian algebras

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Symmetric tensor decomposition and Waring problem (1770)

Symmetric tensor decomposition problem:

Given a homogeneous polynomial $F \in \mathcal{S}_d$ of degree d in the variables $\underline{\mathbf{x}} = (x_0, x_1, \ldots, x_n)$ with coefficients $\in \mathbb{K}$:

$$F(\overline{\mathbf{x}}) = \sum_{|\alpha|=d} F_\alpha \, \underline{\mathbf{x}}^\alpha,$$

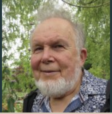
find a minimal decomposition of F of the form

$$F(\underline{\mathbf{x}}) = \sum_{i=1}^{r} \omega_i (\xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d$$

with $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n}) \in \overline{\mathbb{K}}^{n+1}$ spanning disctint lines, $\omega_i \in \overline{\mathbb{K}}$.

The minimal r in such a decomposition is called the **rank** of T.

Generalized Additive or Iarrobino's Decomposition



Generalized Additive Decomposition problem:

find r', $w_i(\underline{\boldsymbol{x}}) \in \mathcal{S}_{k_i}$ for $i = 1, \ldots, r'$ and $\Xi = [\xi_1, \ldots, \xi_{r'}] \in \mathbb{K}^{(n+1)\times r'}$ such that

$$F = \sum_{i=1}^{r'} \omega_i(\underline{\boldsymbol{x}})\,(\xi_i, \underline{\boldsymbol{x}})^{d-k_i}$$

with $\mathrm{rank}_{GAD}(F) = \sum_{i=1}^{r'} \dim\langle\langle \omega_i \rangle\rangle_{\xi_i}$ **minimal**, where

$$\langle\langle \omega_i \rangle\rangle_{\xi_i} = \langle (\xi_i, \underline{\boldsymbol{x}})^{\alpha_1+\cdots+\alpha_n} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}(\omega_i), \alpha_i \in \mathbb{N} \rangle$$

for a basis of $\{(\xi_i, \underline{\boldsymbol{x}}), x_1, \ldots, x_n\}$ of $\mathcal{S}_1$

Example: For $d > 5$, $F = x_0^{d-1} x_1 + (x_0 + x_1 + 2\,x_2)^{d-2}(x_0 - x_1)^2$ is a GAD of

$$\begin{aligned} \mathrm{rank}_{gad}(F) &= \dim\langle\langle x_1 \rangle\rangle_{\xi_1} + \dim\langle\langle (x_0 - x_1)^2 \rangle\rangle_{\xi_2} \\ &= \dim\langle x_1, \ell_1 \rangle + \dim\langle (x_0 - x_1)^2, 2\,\ell_2 (x_0 - x_1), \ell_2^2 \rangle = 5 \end{aligned}$$

with $\xi_1 = [1, 0, 0]$, $\xi_2 = [1, 1, 2]$, $\ell_1 = (\xi_1, \underline{\boldsymbol{x}})$, $\ell_2 = (\xi_2, \underline{\boldsymbol{x}})$.

Geometric point of view

- $\mathcal{V}_{n+1,d} = \{\omega\,(\xi, \underline{\boldsymbol{x}})^d, \omega \in \mathbb{K}, \xi \in \mathbb{K}^{n+1}, \xi \neq 0\}$ **Veronese** variety
- $\mathcal{T}_{n+1,d} = \{\omega(\underline{\boldsymbol{x}})\,(\xi, \underline{\boldsymbol{x}})^{d-1}, \omega(\underline{\boldsymbol{x}}) \in \mathcal{S}_1, \xi \in \mathbb{K}^{n+1}, \xi \neq 0\}$ **tangential** variety (= points on tangents to $\mathcal{V}_{n+1,d}$).
- $\mathcal{O}^k_{n+1,d} = \{\omega(\underline{\boldsymbol{x}})\,(\xi, \underline{\boldsymbol{x}})^{d-k}, \omega(\underline{\boldsymbol{x}}) \in \mathcal{S}_k, \xi \in \mathbb{K}^{n+1}, \xi \neq 0\}$ **osculating** variety (= points on oscullating linear spaces to $\mathcal{V}_{n+1,d}$).

Proposition

The singular locus of $\mathcal{O}^k_{n+1,d}$ is $\mathcal{O}^{k-1}_{n+1,d}$, $\mathcal{V}_{n+1,d} = \mathcal{O}^0_{n+1,d}$ is smooth.

$$F = \sum_{i=1}^{r'} \omega_i(\underline{\boldsymbol{x}})(\xi_i, \underline{\boldsymbol{x}})^{d-k_i} \quad \text{iff} \quad F \in \sum_{i=1}^{r'} \mathcal{O}^{k_i}_{n+1,d}$$

☞ Find the decomposition by associating to F an **Artinian algebra** $\mathcal{A}$

☞ Using Macaulay correspondence, describe $\mathcal{A}^*$ via **inverse systems**

☞ Use **apolar duality** to associate to $F \in \mathcal{S}_d$ an element $F^* \in \mathcal{S}_d^*$:

Apolar product: For $F = \sum_{|\alpha|=d} F_\alpha \underline{\boldsymbol{x}}^\alpha$, $F' = \sum_{|\alpha|=d} F'_\alpha \underline{\boldsymbol{x}}^\alpha \in \mathcal{S}_d$,

$$\langle F, F' \rangle_d = \sum_{|\alpha|=d} \binom{d}{\alpha}^{-1} F_\alpha F'_\alpha.$$

Apolar duality: For $F \in \mathcal{S}_d$ and $q \in \mathcal{S}_k$,

- $F^* : p \in \mathcal{S}_d \mapsto \langle F, p \rangle_d \in \mathbb{K}$ is a linear functional $\in \mathcal{S}_d^*$
- $q \star F^* : p \in \mathcal{S}_{d-k} \mapsto \langle F, q\, p \rangle_d$

☞ Use **Catalecticant, Hankel** operator H_F to describe $\mathcal{A}^*$ as $\mathrm{Im}\, H_F$:

$$\begin{aligned} H_F^{k,d-k} : \mathcal{S}_{d-k} &\rightarrow S_k^* \\ q &\mapsto q \star F^* \end{aligned}$$

Matrix form: for $A \subset \mathcal{S}_k$, $B \subset \mathcal{S}_{d-k}$, $H_F^{A,B} = [\langle F, a\, b \rangle_d]_{a \in A, b \in B}$

Duality (char $\mathbb{K} = 0$)

$\mathcal{S} = \mathbb{K}[x_0, \ldots, x_n] = \mathbb{K}[\overline{x}]$, $\mathcal{S}_d = \mathbb{K}[\overline{x}]_d$, $R = \mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[\boldsymbol{x}]$,
$R_{\leq d} := \mathbb{K}[\boldsymbol{x}]_{\leq d} = \langle \boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ **with** $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq d \rangle$.

- **Linear functionals:** $\Lambda \in R^* = \{\Lambda : R \to \mathbb{K},$ **linear**$\} =$ **Hom**$_{\mathbb{K}}(R, \mathbb{K})$

$$\Lambda : p = \sum_\alpha p_\alpha x^\alpha \mapsto \langle \Lambda | p \rangle = \sum_\alpha \Lambda_\alpha p_\alpha$$

The coefficients $\langle \Lambda | x^\alpha \rangle = \Lambda_\alpha \in \mathbb{K}$, $\alpha \in \mathbb{N}^n$ are called the **moments** of Λ.

- **Formal power series:** $\Lambda(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \Lambda_\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!} \in \mathbb{K}[[z_1, \ldots, z_n]]$

  where $\alpha! = \prod \alpha_i!$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}$.

  $(\frac{1}{\alpha!} \boldsymbol{z}^\alpha)_{\alpha \in \mathbb{N}^n}$ dual basis in R^* of the monomial basis $(\boldsymbol{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ of R.

  $\boldsymbol{z}^\alpha : p \in R \mapsto \partial^\alpha(p)(0)$

- **Truncated linear functionals:** $R^*_{\leq d} = \{\Lambda : R_{\leq d} \to \mathbb{K},$ linear $\}$.

- **Truncated series:**

  For $\Lambda(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \Lambda_\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!} \in \mathbb{K}[[\boldsymbol{z}]]$, $\Lambda(\boldsymbol{z})^{[\leq d]} = \sum_{|\alpha| \leq d} \Lambda_\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!} \in \mathbb{K}[\boldsymbol{z}]_{\leq d} = R^*_{\leq d}$.

From tensors to truncated linear functionals

Let $h_{d,x_0} : p \in R_{\leq d} \mapsto x_0^d p(\frac{\mathbf{x}}{x_0}) \in \mathcal{S}_d$ and

$$\check{F} = F^* \circ h_{d,x_0} \in R^*_{\leq d}$$

- $F = \sum_{|\alpha|=d} F_\alpha x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in S_d \Rightarrow \check{F} = \sum_{|\alpha|=d} F_\alpha \frac{1}{\alpha!} z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in R^*_{\leq d}$

- $F = (\bar{\xi}, \underline{\mathbf{x}})^d$ with $\bar{\xi}_0 = 1 \Rightarrow \check{F} = (\mathbf{e}_{\xi_1,\ldots\xi_n}(\mathbf{z}))^{[\leq d]}$

where $\mathbf{e}_\xi(\mathbf{z}) = \sum_{\alpha \in \mathbb{N}^n} \xi^\alpha \frac{\mathbf{z}^\alpha}{\alpha!}$ is the **evaluation** linear functional $\mathbf{e}_\xi : p \in R \mapsto p(\xi)$.

- $F = \omega(\underline{\mathbf{x}}) L_0(\underline{\mathbf{x}})^{d-k}$ with $L_0 = (\bar{\xi}, \underline{\mathbf{x}})$, $\bar{\xi}_0 = 1$,

$\omega(\underline{\mathbf{x}}) = \omega_0 L_0^k + \omega_1(\mathbf{x}) L_0^{k-1} + \cdots + \omega_k(\mathbf{x}) \in \mathcal{S}_k$, $\omega_i \in \mathcal{S}_i(x_1, \ldots, x_n) \Rightarrow$

$$\check{F} = (\check{\omega}(\mathbf{z})\, \mathbf{e}_\xi(\mathbf{z}))^{[\leq d]}$$

where $\xi = (\bar{\xi}_1, \ldots, \bar{\xi}_n) \in \mathbb{K}^n$, $\check{\omega}(\mathbf{z}) = \sum_{i=0}^k \omega_i(\mathbf{z}) \in \mathbb{K}[\mathbf{z}]_{\leq k}$.

Structure of $\mathcal{A}$

Theorem:

If $I = Q_1 \cap \cdots \cap Q_{r'}$ with Q_i $\boldsymbol{m}_{\xi_i}$-primary, then

- $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_{r'}\}$
- $\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_{r'}$ with $\mathcal{A}_i = \mathcal{R}/Q_i$
- $1 = \boldsymbol{u}_1 \oplus \cdots \oplus \boldsymbol{u}_{r'}$ with $\mathcal{A}_i = \boldsymbol{u}_i\, \mathcal{A}$, $\boldsymbol{u}_i^2 = \boldsymbol{u}_i$, $\boldsymbol{u}_i \boldsymbol{u}_j = 0$ if $i \neq j$.
  ($\boldsymbol{u}_i$ idempotents).

Theorem:

In a basis of $\mathcal{A}$, all the matrices $M_g : a \in \mathcal{A} \mapsto g\, a \in \mathcal{A}$ ($g \in \mathcal{A}$) are of the form

$$\mathtt{M}_g = \begin{bmatrix} \mathtt{M}_g^{\mathbf{1}} & & 0 \\ & \ddots & \\ 0 & & \mathtt{M}_g^{r'} \end{bmatrix} \text{ with } \mathtt{M}_g^{i} = \begin{bmatrix} g(\xi_i) & \star & \star \\ & \ddots & \star \\ 0 & & g(\xi_i) \end{bmatrix}$$

Corollary (Chow form)

$\Delta(\boldsymbol{u}) = \det(v_0 + v_1\, \mathtt{M}_{x_1} + \cdots + v_n\, \mathtt{M}_{x_n}) = \prod_{i=1}^{r} (v_0 + v_1 \xi_{i,1} + \cdots + v_n \xi_{i,n})^{\mu_{\xi_i}}$ *where* $\mu_{\xi_i} = \dim \mathcal{A}_i$ *is the multiplicity of* ξ.

Structure of the dual $\mathcal{A}^*$

Definition (Polynomial-Exponential series)

$$\mathcal{P}ol\mathcal{E}xp = \left\{ \sigma(\mathbf{z}) = \sum_{i=1}^{r} \omega_i(\mathbf{z})\, \mathbf{e}_{\xi_i}(\mathbf{z}) \mid \omega_i(\mathbf{z}) \in \mathbb{K}[\mathbf{z}], \xi_i \in \mathbb{K}^n \right\}$$

where $\mathbf{e}_{\xi_i}(\mathbf{z}) = e^{z_1\xi_{i,1}+\cdots+z_n\xi_{i,n}} = \sum_\alpha \xi_i^\alpha \frac{\mathbf{z}^\alpha}{\alpha!}$ *is* the evaluation $\mathbf{e}_\xi : p \in R \mapsto p(\xi)$.

Theorem:

For $\mathbb{K} = \overline{\mathbb{K}}$ algebraically closed and $\mathcal{A} = \mathcal{R}/I$ artinian with $I = Q_1 \cap \cdots \cap Q_{r'}$, Q_i $\mathbf{m}_{\xi_i}$-primary,

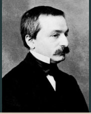
$$\mathcal{A}^* = I^\perp = \oplus_{i=1}^{r'} \mathcal{D}_i\, \mathbf{e}_{\xi_i}(\mathbf{z}) \quad \subset \quad \mathcal{P}ol\mathcal{E}xp$$

- $\mathcal{D}_i = Q_i^\perp = \langle\langle \omega_{i,1}(\mathbf{z}), \ldots, \omega_{i,l_i}(\mathbf{z}) \rangle\rangle$ with $\omega_{i,j}(\mathbf{z}) \in \mathbb{K}[\mathbf{z}]$.
- $\dim_{\mathbb{K}}(\mathcal{D}_i) = \mu_i$ multiplicity of ξ_i.

where $\mathcal{D}_i$ is the **Inverse system** generated by $\omega_{i,1}(\mathbf{z}), \ldots, \omega_{i,l_i}(\mathbf{z}) \in \mathbb{K}[\mathbf{z}]$

$$\langle\langle \omega_{i,1}(\mathbf{z}), \ldots, \omega_{i,l_i}(\mathbf{z}) \rangle\rangle = \langle \partial_{\mathbf{z}}^\alpha(\omega_{i,j}), \alpha \in \mathbb{N}^n, j = 1, \ldots, l_i \rangle$$

Kronecker theorem (Univariate Series)



Kronecker (1881)

The Hankel operator

$$\begin{array}{rcl} H_\Lambda : \mathbb{C}^{\mathbb{N},finite} & \rightarrow & \mathbb{C}^{\mathbb{N}} \\ (p_m) & \mapsto & (\sum_m \Lambda_{m+n} p_m)_{n\in\mathbb{N}} \end{array}$$

is of **finite rank** r iff $\exists \omega_1, \ldots, \omega_{r'} \in \mathbb{C}[z]$ and $\xi_1, \ldots, \xi_{r'} \in \mathbb{C}$ distincts s.t.

$$\Lambda(z) = \sum_{n\in\mathbb{N}} \Lambda_n \frac{z^n}{n!} = \sum_{i=1}^{r'} \omega_i(z) \boldsymbol{e}_{\xi_i}(z)$$

with $\sum_{i=1}^{r'} (\deg(\omega_i) + 1) = r$.

Generalized Kronecker Theorem

Theorem:

For $\Lambda \in R^*$, the Hankel operator

$$\begin{aligned} H_\Lambda : R &\to R^* \\ p &\mapsto p \star \Lambda \end{aligned}$$

is of rank r iff

$$\Lambda(\boldsymbol{z}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z}) \quad \text{with } \omega_i(\boldsymbol{z}) \in \mathbb{K}[\boldsymbol{z}],$$

with $r = \sum_{i=1}^{r'} \dim\langle\langle \omega_i(\boldsymbol{z}) \rangle\rangle = \sum_{i=1}^{r'} \dim\langle \partial_{\boldsymbol{z}}^{\gamma} \omega_i(\boldsymbol{z}) \rangle$. In this case, we have

- $I_\Lambda = \ker H_\Lambda$ with $\mathcal{V}_{\mathbb{C}}(I_\Lambda) = \{\xi_1, \ldots, \xi_{r'}\}$.
- $I_\Lambda = Q_1 \cap \cdots \cap Q_{r'}$ with $Q_i^\perp = \langle\langle \omega_i \rangle\rangle\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$.

C.f. [M'2018]

☞ $\mathcal{A}_\Lambda$ is **Gorenstein**:$(a, b) \mapsto \langle \Lambda | ab \rangle$ is non-degenerate in $\mathcal{A}_\Lambda$.

☞ Can be generalized to $\Lambda = (\Lambda_1, \ldots, \Lambda_m) \in (R^*)^m$.

For $F = \sum_{i=1}^{r} \omega_i(\underline{\boldsymbol{x}})(\overline{\xi}_i, \underline{\boldsymbol{x}})^d$ with $\overline{\xi}_{i,0} = 1$,

- $I_F := \{p \in R \text{ s.t. } \theta(\partial)p(\xi_i) = 0 \text{ for } \theta(\boldsymbol{z}) \in \langle\langle \breve{\omega}_i \rangle\rangle\}$
- $\mathcal{A}_F := R/I_F$ the quotient algebra by I_F.

Theorem:

Let $A = \{a_1, \ldots, a_s\}, A' = \{a'_1, \ldots, a'_t\}$, $B \subset A, B' \subset A'$ s.t.
$B^+ = B \cup x_1 B \cup \cdots \cup x_n B \subset A$, $B'^+ \subset A'$ and $H = H_F^{A',A}$.
If B and B' are bases of $\mathcal{A}_F$, then

- $\ker H = I_F \cap \langle A \rangle$ and $I_F = (\ker H)$
- $\operatorname{im} H = (I_F^\perp)_{|\langle A' \rangle}$ where $I_F^\perp = \{\Lambda \in \mathbb{K}[\boldsymbol{x}]^* \mid \forall p \in I_F, \langle \Lambda, p \rangle = 0\} = \mathcal{A}_F^*$
- $\mathcal{A}_F = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_{r'}$ with $\mathcal{A}_i^* = \langle\langle \breve{\omega}_i(\boldsymbol{z}) \rangle\rangle \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$
- $\mathrm{rank}_{gad}(F) = \mathrm{rank}(H_{\check{F}}^{A',A}) = r = \mu_1 + \cdots \mu_{r'}$ where $\mu_i = \dim \langle\langle \breve{\omega}_i(\boldsymbol{z}) \rangle\rangle$
- $H_0 = H_{\check{F}}^{B',B}$ is **invertible**.
- For $H_i = H_{\check{F}}^{B',x_i B}$, $M_i = H_0^{-1} H_i$ = **multiplication** by x_i in the basis B of $\mathcal{A}_F$.

☞ Decomposition via Schur factorization of the M_i.

Example (Joint work with E. Barilli, D. Taufer)

Let us take $F = x_0^3 x_1 x_2 + (x_0 + 0.5x_1 + 2.0x_2)^4 (x_0 + x_2)$.

- Compute $H = H_F^{2,3}$ of size 6×10 of rank 6.
- $H_0 = H^{B,\ell_0 B}$ invertible where $B = \{x_0^2, x_0x_1, x_0x_2, x_1^2 x_{1,x_2}, x_2^2\}$, ℓ_0 random in $\mathcal{S}_1$. We dedcuce $M_k = H_0^{-1} H^{B,x_k B}$, $k = 0, \ldots, 2$
- Compute a Schur Factorization $M_{rnd} = QTQ^t$ of $M_{rnd} = \sum_k \lambda_k M_k$
- Deduce blocks $M_k^{[1]}$ of size 2×2, $M_k^{[j]}$ of size 4×4 of local multiplication by $\frac{x_k}{\ell_0}$ from $Q^t M_i Q$ and the associated points or linear forms:

$$\begin{aligned} \ell_1 &= 26.43x_2 + 6.61x_1 + 13.21x_0, \\ \ell_2 &= -3.76 \times 10^{-14} x_2 - 2.04 \times 10^{-15} x_1 - 3.62x_0; \end{aligned}$$

- Compute the nil-index 2 (resp. 3) of $[M_1^{[j]} - \xi_{i,j}]_{i=1,2}$ and deduce the degree 1 (resp. 2) of ω_k and solve $F = \omega_1 \ell_1^4 + \omega_2 \ell_2^3$ to get

$$\begin{aligned} \omega_1 &= 3.2810 \times 10^{-5}(x_0 + x_2) - 1.7949 \times 10^{-19} x_1, \\ \omega_2 &= -1.5305 \times 10^{-15} x_2^2 - 0.0211 x_1 x_2 - 8.6353 \times 10^{-18} x_1^2 \\ &\quad - 7.7109 \times 10^{-16} x_0 x_2 - 9.0234 \times 10^{-17} x_0 x_1 - 1.2588 \times 10^{-16} x_0^2; \end{aligned}$$

We get $\|F - T\| \approx 7.60 \times 10^{-14}$ where $T = \sum_{i=1}^{2} \omega_i \cdot \ell_i^{d-deg(w_i)}$

Definition (Truncated Normal Form)

Let $W \subset R = \mathbb{K}[\boldsymbol{x}]$ and V be a $\mathbb{K}$-vector space. A **Truncated Normal Form (TNF)** for the ideal I from W to V is a linear map $\boldsymbol{N} : W \to V$ s.t.

$$0 \to K \to W \xrightarrow{\boldsymbol{N}} V \to 0$$

is exact, $I = (\ker \boldsymbol{N})$, $I \cap W = \ker \boldsymbol{N}$ and $I + W = R$.

If $\dim V = r$ and $\boldsymbol{N}$ is a TNF then $\dim \mathcal{A} = R/I = r$ and $\boldsymbol{N} = \mathcal{N}_{|W}$ where $\operatorname{Im} \mathcal{N}^t = I^\perp$ (i.e. $\mathcal{N}$ is a **Normal Form** for I).

TNF from tensor or (truncated) linear functionals $\Lambda \in R^*$:

Theorem:

Let $B \subset A$, $B' \subset A'$ stably connected[a], $B^+ \subset A$, $B^+ \subset A'$ with

- $H_\Lambda^{B',B}$ **invertible** and
- $\mathrm{rank}\, H_\Lambda^{A',A} = \mathrm{rank}\, H_\Lambda^{B',B}$ (**flat extension**).

Then $\boldsymbol{N} := H_\Lambda^{B',A}$ is a **TNF** from $\langle A \rangle$ to $\langle B' \rangle^*$.

[a] If $m \in A$, then $m = 1$ or $\exists i_m \in [n]$, $m' \in A$ s.t. $m = x_{i_m} m'$.

C.f. [BCMT 2010]

Definition (Operators of multiplication)

For a map $\boldsymbol{N} : W \to V$ such that there exists $B \subset W$ with $B^+ \subset W$, $\boldsymbol{N}_{|B} : B \to V$ bijective, we define the multiplication operators associated to $\boldsymbol{N}$ as

$$\begin{aligned} \boldsymbol{M}_k : B &\to B \\ b &\mapsto (\boldsymbol{N}_{|B})^{-1} \circ \boldsymbol{N}(x_k b). \end{aligned}$$

Theorem:

Let $\boldsymbol{N} : W \to V$ and $B \subset W \subset R$ stably connected s.t. $B^+ \subset W$ and $\boldsymbol{N}_{|B} : B \to V$ is an isomorphism. Then

$\boldsymbol{N}$ is a TNF for $(\ker \boldsymbol{N}_{|B^+})$ from B^+ to V $\Leftrightarrow$ $\boldsymbol{M}_k$ pairewise commute.

C.f. [M'99]

The associated Normal Form is

$$\begin{aligned} \mathcal{N} : R &\longrightarrow \langle B \rangle \\ p(\boldsymbol{x}) &\longmapsto p(\boldsymbol{M})(1). \end{aligned}$$

$(\boldsymbol{M}, 1)$ a.k.a. stable *Atiyah-Drinfel'd-Hitchin-Mani (ADHM) datum.*

Hilbert Scheme

For $d \geq \rho$ (Gotzmann regularity), let $\boldsymbol{N} : \mathcal{S}_d \to V$ with $\operatorname{rank} \boldsymbol{N} = \dim V = r$, then $\boldsymbol{N} \sim \Gamma \in \mathrm{Gr}_r(\mathcal{S}_d^*)$.

Its **Plücker coordinates** are

$$\Delta_B = \det \begin{array}{c} \\ \Lambda_1 \\ \vdots \\ \Lambda_r \end{array} \begin{array}{c} \begin{array}{ccc} \underline{\boldsymbol{x}}^{\beta_1} & \cdots & \underline{\boldsymbol{x}}^{\beta_r} \end{array} \\ \begin{bmatrix} \boldsymbol{N}_{1,\beta_1} & \dots & \boldsymbol{N}_{1,\beta_r} \\ \vdots & & \vdots \\ \boldsymbol{N}_{r,\beta_1} & \dots & \boldsymbol{N}_{r,\beta_r} \end{bmatrix} \end{array} \quad \text{for } B = \{\underline{\boldsymbol{x}}^{\beta_1}, \dots, \underline{\boldsymbol{x}}^{\beta_r}\} \subset \mathcal{S}_d$$

Definition (Hilbert Scheme of r points)

$\mathrm{Hilb}^r = \{\boldsymbol{N} \in \mathrm{Gr}_r(\mathcal{S}_d^*) \text{ s.t. } \boldsymbol{N} = I_d^{\perp}, \dim \mathcal{A} = \dim R/\bar{I} = r\}$

- B bais of $\mathcal{S}_d/I_d$ iff $\Delta_B \neq 0$
- Normal form of $c = \underline{\boldsymbol{x}}^{\alpha} \in \mathcal{S}_d$ on B: $\boldsymbol{N}(c) = [\frac{\Delta_{B[b \to c]}}{\Delta_B}]_{b \in B}$

Theorem:

Let $d \geq r$. $\boldsymbol{N} \in \mathrm{Gr}_r(\mathcal{S}_d^*)$ is in the Hilbert scheme of r points iff for all subset B of r monomials of degree $d-1$, for all $K \in \mathbb{N}^{r+1}$ with $|K| = 2\,r$, for all $b, b'' \in B$ and all $i \leq j$, we have

$$\sum_{I+J=K} \sum_{b' \in B} \Big(\Delta_{\underline{\boldsymbol{x}}_I \circ B^{[x_{I_{b'}}\, b' \to x_i b]}} \, \Delta_{\underline{\boldsymbol{x}}_J \circ B^{[x_{J_{b''}}\, b'' \to x_j b']}} - \Delta_{\underline{\boldsymbol{x}}_I \circ B^{[x_{I_{b'}}\, b' \to x_j b]}} \, \Delta_{\underline{\boldsymbol{x}}_J \circ B^{[x_{J_{b''}}\, b'' \to x_i b']}}\Big) = 0 \quad (1)$$

$$\sum_{I+J=K} \left(\Delta_{\underline{\boldsymbol{x}}_I \circ B^{[x_{I_b}\, b \to x_k a]}} \, \Delta_{J.B} \; - \sum_{b' \in B} \Delta_{\underline{\boldsymbol{x}}_I \circ B^{[x_{I_b}\, b \to x_k b']}} \, \Delta_{\underline{\boldsymbol{x}}_J \circ B^{[x_{J_{b'}}\, b' \to x_{J_{b'}}\, a]}} \right) = 0 \quad (2)$$

(1) = **commutation relations** for a basis B

(2) = **rewriting rules** of the monomials of $\mathcal{S}^d$

[Alonso-Brachat-M'10]

Thanks for your attention.




Happy birthday Tony !

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