Hilbert functions of Veronesean subvarieties and Complete Intersections

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A long-standing (and solved) problem

Problem (G. Cramer, L. Euler, 1744)

Let X a set of points in the plane. When is X the intersection of two curves?

Theorem (L. Euler, 1744)

If a set $X \subset \mathbb{P}^2$ of 9 points is the intersection of two cubics, then every cubic passing through 8 of the 9 passes through all 9.

The viceversa is true, provided that **the 9 points do not lie on a conic**.

Theorem (E. Davis, P. Maroscia, 1984)

Let $X \subset \mathbb{P}^2$ a set of ℓ points and set $\alpha \coloneqq \min\{t \in \mathbb{N} \mid h^0(\mathcal{I}_X(t)) \neq 0\}$. Then, X is a complete intersection if and only if $\alpha^{-1}\ell \in \mathbb{N}$ and $H^0(\mathcal{I}_X(t)) = H^0(\mathcal{I}_Y(t))$ for any $Y \subset X$ with $\ell(Y) = \ell - 1$ and any $t \leq \alpha^{-1}\ell + \alpha - 3$.

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What about complete intersection on Veronese surfaces? We classify them using Hilbert functions.

Basic ingredients

- $\mathbb{K}$ is an algebraic closed field, $\mathrm{char}(\mathbb{K}) = 0$.
- Our varieties will always be reduced.
- $R = \mathbb{K}[x_0, \dots, x_n]$ is the coordinate ring of $\mathbb{P}^n$.
- If d is a positive integer, we set $N = \binom{n+d}{d} - 1$ and we denote the coordinate ring of $\mathbb{P}^N$ by $S = \mathbb{K}[y_0, \dots, y_N]$.
- $H_{\mathbb{X}}(t) := \dim_{\mathbb{K}}(R/\mathcal{I}(\mathbb{X}))_t$ is the Hilbert function of a projective variety $\mathbb{X}$ and $\Delta H_{\mathbb{X}}(t) = H_{\mathbb{X}}(t) - H_{\mathbb{X}}(t-1)$ is its first difference function.
- for each $n, d \in \mathbb{N}_{>0}$ we denote by $\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ the (n,d)-Veronese embedding and by $V_{n,d} \coloneqq \nu_{n,d}(\mathbb{P}^n)$.

Hilbert functions of Veronese subvarieties

Using the graded morphism

$$\begin{array}{rccc} \varphi_d : & S & \to & R \\ & a & \mapsto & a \\ & y_i & \mapsto & \underline{x}^{\alpha_i} \end{array}$$

for all $a \in \mathbb{K}$, and for $i \in \{0, \dots, N\}$, it is easy to see that:

1. if $\mathbb{X} \subseteq V_{n,d}$, then $(\mathcal{I}(\nu_{n,d}^{-1}(\mathbb{X})))_{td} = \varphi_d(\mathcal{I}(\mathbb{X})_t)$.
2. if $\mathbb{X}$ is a subvariety of $V_{n,d}$ and we set $\mathbb{Y} = \nu_{n,d}^{-1}(\mathbb{X})$, then

$$H_{\mathbb{X}}(t) = H_{\mathbb{Y}}(td) \; \forall \; t \geq 0.$$

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$$H_{\mathbb{X}}(t) = H_{\mathbb{Y}}(td) \ \forall\, t \geq 0.$$

Theorem (-, E. Carlini '23)

Let $h(t): \mathbb{N} \to \mathbb{N}$ be the Hilbert function of a projective variety $\mathbb{X} \subseteq \mathbb{P}^N$. Then there exists $\mathbb{X}' \subseteq V_{n,d} \subseteq \mathbb{P}^N$ such that $H_{\mathbb{X}'}(t) = h(t)$ if and only there exists $k(t): \mathbb{N} \to \mathbb{N}$ Hilbert function of a projective variety in $\mathbb{P}^n$ such that $h(t) = k(dt)$.

0-sequences

If h and i are positive integers, then h can be written uniquely in the form

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. We set

$$h^{<i>} = \binom{n_i + 1}{i+1} + \binom{n_{i-1} + 1}{i} + \cdots + \binom{n_j + 1}{j+1}$$

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Definition

A sequence of non-negative integers $(c_t)_{t\in\mathbb{N}}$ is called a 0-*sequence* if $c_0 = 1$ and $c_{t+1} \leq c^{\langle t\rangle}$ for all $t \geq 1$.

Theorem (Macaulay '27, Stanley, '78)

The following two are equivalent (for $\mathbb{K}$ any field):

- $(c_t)_{t\in\mathbb{N}}$ is the Hilbert function of a standard algebra $\mathbb{K}[x_0, \ldots, x_n]/I$.
- $(c_t)_{t\in\mathbb{N}}$ is a 0-sequence.

Differentiable 0-sequences

If $I \subseteq \mathbb{K}[x_0, \ldots, x_n]$ is a radical ideal, there exists a linear form $L \in \mathbb{K}[x_0, \ldots, x_n]_1$ such that the following sequence

$$0 \longrightarrow R/_{I}(-1) \xrightarrow{L} R/_{I} \xrightarrow{\pi} R\Big/_{I+(L)} \longrightarrow 0$$

is exact and hence

$$\Delta H_I(t) = H_{I+(L)}(t)$$

Definition

A 0-sequence $(b_t)_{t \in \mathbb{N}}$ is called *differentiable* if the difference sequence $(c_t)_{t \in \mathbb{N}}$, $c_t = b_t - b_{t-1}$ is again a 0-sequence (where $b_{-1} = 0$).

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Theorem (A.V. Geramita, P. Maroscia, L. Robert, '83)

Let $\mathbb{K}$ be an infinite field. The following two are equivalent:

- $(b_t)_{t\in\mathbb{N}}$ is the Hilbert function of $\mathbb{K}[x_0, \ldots, x_n]/I$, with $I = \sqrt{I}$.
- $(b_t)_{t\in\mathbb{N}}$ is a differentiable 0-sequence.

From 0-sequences to d-sequences

Definition (-, E. Carlini '23)

- A 0-sequence $(b_t)_{t\in\mathbb{N}}$ is called *d-sequence* if there exists a 0-sequence $(c_t)_{t\in\mathbb{N}}$ such that $b_t = c_{(d+1)t}$.
- A 0-sequence $(b_t)_{t\in\mathbb{N}}$ is called *differentiable d-sequence* if there exists a differentiable 0-sequence $(c_t)_{t\in\mathbb{N}}$ such that $b_t = c_{(d+1)t}$.

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We can now rephrase our theorem as follows.

Theorem (-, E. Carlini '23)

Let $(h_t)_{t\in\mathbb{N}}$ be a sequence of non-negative integers such that $h_0 = 1$ and $h_1 = N+1$. There exists a projective variety $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^N$ such that $H_{\mathbb{X}}(t) = h_t$ if and only if $(h_t)_{t\in\mathbb{N}}$ is a differentiable $(d-1)$-sequence.

Castelnuovo functions

Proposition (P. Dubreil '33)

Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a 0-dimensional scheme and let $\alpha_{\mathbb{X}} := \min\{t \in \mathbb{N} \mid (\mathcal{I}(\mathbb{X})_t \neq 0)\}$. Then there exists $\sigma_{\mathbb{X}} \in \mathbb{N}$, $\sigma_{\mathbb{X}} \geq \alpha_{\mathbb{X}} - 2$ such that:

1. $\Delta H_{\mathbb{X}}(t) = 0$ for $t < 0$.
2. $\Delta H_{\mathbb{X}}(t) = t + 1$ if and only if $t = 0, \ldots, \alpha_{\mathbb{X}} - 1$.
3. $\Delta H_{\mathbb{X}}(t) \geq \Delta H_{\mathbb{X}}(t+1)$ for $\alpha_{\mathbb{X}} - 1 \leq t \leq \sigma_{\mathbb{X}} + 1$.
4. $\Delta H_{\mathbb{X}}(\sigma_{\mathbb{X}} + 1) > 0$.
5. $\Delta H_{\mathbb{X}}(t) = 0$ for $t > \sigma_{\mathbb{X}} + 1$.

$\Delta H_{\mathbb{X}}(t)$ is the *Castelnuovo function* of $\mathbb{X}$.

Castelnuovo sets

Definition/Theorem (B. Kreuzer, M. Kreuzer, '98)

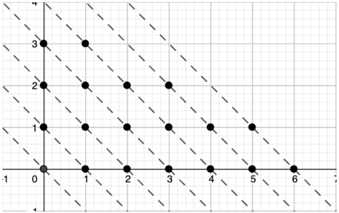
Let $h : \mathbb{Z} \to \mathbb{N}$ be a Castelnuovo function with invariants α_h and σ_h , and let $\{s_0, \dots, s_{\sigma_h+1}\} \subseteq \mathbb{K}, \{t_0, \dots, t_{\alpha_h-1}\} \subseteq \mathbb{K}$ be sets of pairwise distinct elements. The reduced 0-dimensional subscheme

$$\mathbb{X}(h) := \{(1 : s_i : t_j) \in \mathbb{P}^2 \mid 0 \le i + j \le \sigma_h + 1, 0 \le j \le h_{i+j}\}$$

is called *Castelnuovo set* for h with parameters $s_0, \dots, s_{\sigma_h+1}$ and $t_0, \dots, t_{\alpha_h} - 1$. Moreover $\Delta H_{\mathbb{X}}(t) = h(t)$.

An example:

t	0	1	2	3	4	5	6	7
$h(t)$	1	2	3	4	4	3	2	0



The idea in case $n = 2$

Let us suppose to have $h(t)$, the Hilbert function of 128 points in $\mathbb{P}^{20}$ defined by the following table:

t	0	1	2	3	4	5	6
$h(t)$	1	21	62	100	122	128	128

and we want to construct a $k(t) : \mathbb{Z} \to \mathbb{Z}$ such that $h(t) = k(dt)$, i.e. we want to understand if $\exists\, \mathbb{X} \subseteq V_{2,5} \subseteq \mathbb{P}^{20}$ such that $H_{\mathbb{X}}(t) = h(t)$. Since

$$k_t = \sum_{i=0}^{t} \Delta k_i$$

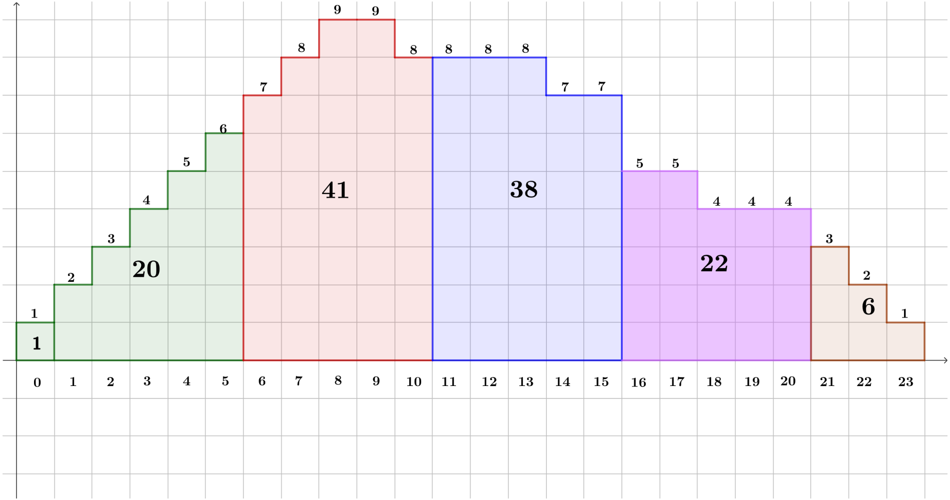
we can construct Δk_t instead of k_t. Moreover, the condition $h(t) = k(dt)$ imposes that

$$\Delta h_{t+1} = \sum_{i=1}^{d} \Delta k_{dt+i}.$$

We have:

t	0	1	2	3	4	5	6
$\Delta h(t)$	1	20	41	38	22	6	0

The idea in case $n = 2$



The functions μ_1 and μ_2

Given $d, t, s \in \mathbb{N}$ such that $s \leq d^2 t + \frac{d(d+3)}{2}$ we define the following two functions:

$$\mu_1(d,t,s) := \underbrace{d^2 t + \frac{d(d+3)}{2}}_{H_{V_{2,d}}(t)} - s$$

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$$\mu_2(d,t,s) := \begin{cases} \left\lfloor \frac{2d(t+1)+3-\sqrt{1+8\mu_1(d,t,s)}}{2} \right\rfloor, & \text{if } 1 \leq \mu_1(d,t,s) \leq \binom{d+1}{2} \\ dt - n, & \text{if } \begin{array}{c} \binom{d+1}{2} + dn < \mu_1(d,t,s) \leq \binom{d+1}{2} + d(n+1) \\ 0 \leq n \leq dt \end{array} \end{cases}$$

Characterization of d-sequences for $n = 2$

Since $n = 2$ we can have just a d-sequence of a curve or of a set of points. The former is already solved since for $n = 2$ curves are divisors, the latter is solved by the following proposition.

Proposition (-, E. Carlini '23)

Let us consider a finite set of reduced points $\mathbb{X} \subseteq \mathbb{P}^{\frac{d(d+3)}{2}}$ and set

$$t_1 = \max\left\{t \mid H_{\mathbb{X}}(t) = H_{V_{2,d}}(t)\right\}, \quad t_2 = \min\left\{t \mid H_{\mathbb{X}}(t) = |\mathbb{X}|\right\}.$$

Then $H_{\mathbb{X}}(t)$ is a differentiable $(d-1)$-sequence if and only if the following conditions hold:

1. $$\mu_2(d, t_1, \Delta H_{\mathbb{X}}(t_1+1)) \geq \left\lceil \frac{\Delta H_{\mathbb{X}}(t_1+2)}{d} \right\rceil;$$
2. For all $t_1 + 2 \leq t \leq t_2 - 1$ $$\left\lfloor \frac{\Delta H_{\mathbb{X}}(t)}{d} \right\rfloor \geq \left\lceil \frac{\Delta H_{\mathbb{X}}(t+1)}{d} \right\rceil.$$

Hilbert functions of reduced points on Veronese surfaces

As an immediate consequence we have the following theorem.

Theorem (-, E. Carlini '23)

Let $(h_t)_{t\in\mathbb{N}}$ be the Hilbert function of a finite set of m reduced points in $\mathbb{P}^{\frac{d(d+3)}{2}}$ and set

$$t_1 = \max\left\{t \mid h(t) = H_{V_{2,d}}(t)\right\} \qquad t_2 = \min\left\{t \mid h(t) = m\right\}.$$

Then there exists $\mathbb{X} \subseteq V_{2,d} \subseteq \mathbb{P}^N$, $|\mathbb{X}| = m$ such that $H_{\mathbb{X}}(t) = h_t$ if and only if the following conditions hold

1. $$\mu_2(d, t_1, \Delta h_{t_1+1}) \geq \left\lceil \frac{\Delta h_{t_1+2}}{d} \right\rceil;$$
2. For all $t_1 + 2 \leq t \leq t_2 - 1$
$$\left\lfloor \frac{\Delta h_t}{d} \right\rfloor \geq \left\lceil \frac{\Delta h_{t+1}}{d} \right\rceil.$$

The proof of the theorem is constructive! (Using Castelnuovo sets)

Example 1

Let us consider the sequence $(h_t)_{t\in\mathbb{N}}$ defined as follows

t	0	1	2	3	4	5	6	7	8	9	10	11
h_t	1	36	120	253	435	666	946	1256	1531	1744	1956	2025

and $h_t = 2025$ for $t \geq 12$. This is the Hilbert function of a set of 2025 reduced points in $\mathbb{P}^{35}$. We ask whether there exists $\mathbb{X} \subseteq V_{2,7} \subseteq \mathbb{P}^{35}$ such that $H_{\mathbb{X}}(t) = h_t$ for all $t \geq 0$. First we determine t_1 and t_2.

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t	0	1	2	3	4	5	6	7	8	9	10	11
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so that $t_1 = 6$ and $t_2 = 11$. To determine $\mu_1(7, 6, \Delta h_{t_1+1})$ we compute Δh_{t_1+1}. We have that

t	0	1	2	3	4	5	6	7	8	9	10	11	12
Δh_t	1	35	84	133	182	231	280	310	275	213	212	69	0

and thus $\mu_1(7, 6, 310) = 7^2 \cdot 6 + \frac{7(7+3)}{2} - 310 = 19$.

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and thus $\mu_1(7, 6, 310) = 7^2 \cdot 6 + \frac{7(7+3)}{2} - 310 = 19$. Finally, since $19 \leq \binom{7+1}{2} = 28$, we get $\mu_2(7, 6, 310) = \left\lfloor \frac{2\cdot 7(6+1)+3-\sqrt{1+8\cdot 19}}{2} \right\rfloor = 44$.

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Let us consider the sequence $(h_t)_{t\in\mathbb{N}}$ defined as follows

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t	0	1	2	3	4	5	6	7	8	9	10	11	12
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t	0	1	2	3	4	5	6	7	8	9	10	11	12
$\left\lceil \frac{\Delta h_t}{7} \right\rceil$	1	5	12	19	26	33	40	45	40	31	31	10	0
$\left\lfloor \frac{\Delta h_t}{7} \right\rfloor$	0	5	12	19	26	33	40	44	39	30	30	9	0

Since $\mu_2(7, 6, 310) = 44$ and $\left\lceil \frac{\Delta h_8}{7} \right\rceil = 40$ condition 1. is satisfied. However condition 2. is not satisfied for $t = 9$ and hence such an $\mathbb{X}$ does not exist.

Example 2

Now, we consider the sequence $(h_t)_{t\in\mathbb{N}}$ defined as follows

t	0	1	2	3	4	5	6	7	8	9	**10**	11
h_t	1	36	120	253	435	666	946	1256	1531	1744	**1915**	2025

and $h_t = 2024$ for $t \geq 12$; note that this function coincides with the one of the previous example, but for $t = 10$. We ask whether there exists $\mathbb{X} \subseteq V_{2,7} \subseteq \mathbb{P}^{35}$ such that $H_{\mathbb{X}}(t) = h_t$ for all $t \geq 0$. As in the previous example we have $t_1 = 6$ and $t_2 = 11$. Moreover, we get

t	0	1	2	3	4	5	6	7	8	9	10	11	12
Δh_t	1	35	84	133	182	231	280	310	275	213	201	110	0
$\left\lceil \frac{\Delta h_t}{7} \right\rceil$	1	5	12	19	26	33	40	45	40	31	29	16	0
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and thus $\mu_1(7, 6, 310) = 19$ and $\mu_2(7, 6, 310) = 44$. Thus, condition 1. is satisfied and condition 2. is satisfied for $t = 8, 9, 10$. Hence such an $\mathbb{X}$ exists.

Complete intersections on Veronese surfaces

Proposition (-, E. Carlini '23)

If $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^N$ is a reduced complete intersection, then $\mathcal{I}(\mathbb{X})$ has a linear generator. Moreover, if $|\mathbb{X}| > 1$, then $\mathcal{I}(\mathbb{X})$ has a quadratic generator.

Theorem (-, E. Carlini '23)

Let $\mathbb{X} \subseteq V_{2,d} \subseteq \mathbb{P}^N$ be a reduced complete intersection. Then $\mathbb{X}$ is one of the following:

1. a reduced point;
2. a set of two reduced points;
3. a conic lying on $V_{2,2} \subset \mathbb{P}^5$;
4. $2b$ points lying on a conic on $V_{2,2} \subset \mathbb{P}^5$ and a hypersurface of degree b.

Complete intersections on $V_{3,2}$

Proposition (-, E. Carlini '23)

Let $\mathbb{X} \subseteq V_{3,2} \subseteq \mathbb{P}^9$ be a reduced complete intersection. Then $\mathbb{X}$ is one of the following:

1. a reduced point;
2. a set of two reduced points;
3. a conic;
4. $2b$ points lying on a conic on $V_{3,2} \subset \mathbb{P}^9$ and a hypersurface of degree b;

Conjecture

Conjecture (-, E. Carlini '23)

Let $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^N$ be a reduced complete intersections with $d > 1$. Then $\mathbb{X}$ is one of the following:

1. a reduced point;
2. a set of two reduced points;
3. a conic lying on $V_{n,2} \subset \mathbb{P}^N$;
4. $2b$ points lying on a conic on $V_{n,2} \subset \mathbb{P}^N$ and a hypersurface of degree b.

Happy birthday, Tony!
Thank you for your attention!