Logarithmic differential forms and questions of residues.

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Contents

I. Basic definitions.

Let $D \subset S = (\mathbb{C}^n, 0)$ be a reduced hypersurface, with equation

$$h = 0$$

We set : $\Theta_S := \mathrm{Der}_{\mathbb{C}}(\mathscr{O}_S) = \mathrm{Hom}_{\mathscr{O}_S}(\Omega^1_S, \mathscr{O}_S)$, $\quad \Omega^p_S(D) = \Omega^p_S \cdot \frac{1}{h}$

Definition (K. Saito)

$$\Omega^p(\log D) := \{\omega \in \Omega^p_S(D) \mid d\omega \in \Omega^{p+1}_S(D)\}$$
$$\mathrm{Der}(-\log D) := \{\delta \in \Theta_S \mid \delta(h) \in \mathscr{I}_D := \mathscr{O}_S \cdot h\}$$

All these modules are coherent and reflexive. In particular

$$\Omega^1(\log D) = \mathrm{Hom}_{\mathscr{O}_S}(\mathrm{Der}(-\log D), \mathscr{O}_S), \text{ and}$$

$$\mathrm{Der}(-\log D) = \mathrm{Hom}_{\mathscr{O}_S}(\Omega^1(\log D), \mathscr{O}_S)$$

Let $\Sigma = Sing(D)$, with $\mathscr{O}_\Sigma = \mathscr{O}_S/(J(h), h) = \mathscr{O}_D/\mathscr{J}_D$. We have the following exact sequence :

$$(1.1) \qquad 0 \longrightarrow \mathrm{Der}(-\log D) \hookrightarrow \Theta_S \xrightarrow{dh} \mathscr{J}_D \longrightarrow 0$$

Definition

The divisor D is free iff $\mathrm{Der}(-\log D)$ or alternatively $\Omega^1(\log D)$ is a free module.

Here are two characterisations of freeness.

Theorem (Saito criterion.)

The divisor D is free iff there are $\delta_1, \cdots, \delta_n \in \mathrm{Der}(-\log D)$ such that

$$\det(\delta_1, \cdots, \delta_n) = uh \quad u \text{ a unit}$$

Theorem (Terao in qh case, Aleksandrov.)

The following three conditions are equivalent :

1. *The divisor D is free.*
2. *$\mathscr{J}_D$ is Cohen Macaulay (maximal as $\mathscr{O}_D$-module.)*
3. *$\mathscr{O}_\Sigma$ is Cohen Macaulay, of dimension $n-2$.*

The proof essentially uses the Auslander-Buchsbaum formula.
NB : A divisor D has Gorenstein singular locus Z of codimension 1 if and only if D is locally the product of a quasihomogeneous plane curve and a smooth space.

II. Logarithmic residues

Saito proves that $\omega \in \Omega^p(\log D)$ iff there is $g \in \mathscr{O}_S$ non zero divisor in $\mathscr{O}_D$ and there are holomorphic forms ξ, η such that :

$$g\omega = \frac{dh}{h} \wedge \xi + \eta,$$

Definition

The residue of ω is the meromorphic $(q-1)$-form on D or equivalently on the normalization $\tilde{D}$:

$$\rho_D^p(\omega) := \frac{\xi}{g}|_D \in \Omega_D^{p-1} \otimes Q(\mathscr{O}_D) = \Omega_{\tilde{D}}^{p-1} \otimes Q(\mathscr{O}_{\tilde{D}})$$

We set $\rho_D^1 = \rho_D$, $\mathscr{R}_D := \rho_D(\Omega^1(\log D)) \subset Q(\mathscr{O}_D)$

Properties of $\mathscr{R}_D$.

Proposition

We have $\mathscr{O}_{\tilde{D}} \subset \mathscr{R}_D$ and there is an exact sequence :

$$(2.1) \qquad 0 \longrightarrow \Omega^1_S \longrightarrow \Omega^1(\log D) \xrightarrow{\rho_D} \mathscr{R}_D \longrightarrow 0.$$

Applying $\mathrm{Hom}(\bullet, \mathscr{O}_S)$ we obtain :

Proposition (G, M. Schulze)

1) There is an exact sequence

$$0 \longrightarrow \mathrm{Der}(-\log D) \longrightarrow \Theta_S \xrightarrow{\sigma_D} \mathscr{R}_D^\vee \longrightarrow \mathrm{Ext}^1_{\mathscr{O}_S}(\Omega^1(\log D), \mathscr{O}_S)$$

2) The image of σ_D is $\mathscr{J}_D \subset \mathscr{R}_D^\vee$ and we always have $\mathscr{R}_D = \mathscr{J}_D^\vee$.
3) When D is free $\mathscr{J}_D = \mathscr{R}_D^\vee$

The presence of the dual residue module $\mathscr{R}_D^\vee$ comes from the *change of ring formula* :

$$\mathscr{R}_D^\vee := \mathrm{Hom}_{\mathscr{O}_D}(\mathscr{R}_D, \mathscr{O}_D) = Ext^1_{\mathscr{O}_S}(\mathscr{R}_D, \mathscr{O}_S)$$

- There is an involution on maximal CM fractional ideals in $Q(\mathscr{O}_D)$, (de Jong and Van Straten). We obtain a chain :

$$\mathscr{J}_D \subseteq \mathscr{R}_D^\vee \subseteq \mathscr{C}_D \subseteq \mathscr{O}_D \subseteq \mathscr{O}_{\widetilde{D}} \subseteq \mathscr{R}_D = \mathscr{J}_D^\vee$$

- If D is free, then $\mathscr{J}_D = \mathscr{R}_D^\vee$ as fractional ideals.
- We call $\mathscr{R}_D = \mathscr{O}_{\widetilde{D}}$ *the normal crossing condition*.

Normal crossing condition.

Theorem (Saito)

For a divisor D in a complex manifold S, consider the following conditions:

1. *the local fundamental groups of the complement $S \backslash D$ are Abelian;*
2. *in codimension one, that is, outside of an analytic subset of codimension at least 2 in D, D is a normal crossing;*
3. *the residue of any logarithmic 1-form along D is a weakly holomorphic function on D.*

Then the implications (A) $\Rightarrow$ (B) $\Rightarrow$ (C) *hold true.*

In his 1980 paper Saito asked for the converse implications:

Theorem (Lê Dung Tràng and K.Saito 1984)

The implication (A) $\Leftarrow$ (B) *in Theorem 3.1 holds true.*

Theorem (G, Mathias Schulze)

The implication (B) $\Leftarrow$ (C) *in Theorem 3.1 holds true: if the residue of any logarithmic* 1*-form along* D *is a weakly holomorphic function on* D *then* D *is a normal crossing in codimension one.*

The proof consists in passing to a nearby point, where D is free.

Characterization of normal crossing divisors

A normal crossing divisor is free and $\mathscr{J}_D$ is a radical ideal of $\mathscr{O}_D$.
A partial converse is :

Theorem (E. Faber, G and M. Schulze.)

For a free divisor with smooth normalization, any of the conditions

- *The ideal* $\mathscr{J}_h = (h'_{x_1}, \cdots, h'_{x_n}) \subset \mathscr{O}_S$ *is a radical ideal*
- *Any of the equivalent conditions* (A), (B), (C),
- *The Jacobian ideal* $\mathscr{J}_D$ *is radical.*

imply that D *is a normal crossing divisor.*

Question (E. Faber) In i) or iii), can one get rid of the smoothness hypothesis?

Three examples.

1) The whitney umbrella $W = \{h = x^2 - y^2 z = 0\}$ satisfies (B). Its normalization is smooth isomorphic to $\mathbb{C}^2$:

$$\mathscr{O}_W \subset \mathbb{C}\{u, y\} = \mathscr{O}_{\widetilde{W}}, \text{ , with } z = u^2, x = uy$$

We find $\Omega^1(\log D) = \mathscr{O}\frac{dh}{h} + \mathscr{O}\omega + \mathscr{O}dy + \mathscr{O}dz$, with $\omega = \frac{yzdx - xzdy - xy/2dz}{h}$, we check that $\mathscr{R}_D = \mathscr{O}_{\widetilde{W}}$ since

$$y\omega = \frac{x}{2}\frac{dh}{h} - dx, \quad \mathrm{Res}(2\omega) = u = \sqrt{z}$$

By this example freeness is necessary in the last theorem.
2) Let $\mathscr{A}$ be a generic hyperplane arrangement. With equation $\prod \ell_i = 0$ then we have

$$\Omega^1(\log A) = \sum \mathscr{O} \cdot \frac{d\ell_i}{\ell_i}$$

3) Let $C \subset \mathbb{P}^2$ be a nodal curve. We should check $\mathscr{R}_X = \mathscr{O}_{\tilde{X}}$.
Example : the cone on the cubic nodal curve.

$$h := y^2 z - x^2(z+x) = 0.$$

$$\mathscr{O}_X = \mathbb{C}[x+z, y, z] = \mathbb{C}[t^2u, t^3 - tu^2, u^3] \subset \mathscr{O}_{\tilde{X}} = \mathbb{C}[t^3, t^2u, tu^2, u^3],$$

with $t^2u = x + z$, $tu^2 = \frac{yz}{x}$. We find $tu^2 = \mathrm{Res}(\alpha/h)$,

$$\alpha = -(2z+3x)yzdx + 2xz(z+x)dy + x^2ydz$$

$$\text{Because } x\alpha = yz \cdot dh \quad (\mod f\Omega^1_{\mathbb{C}^3})$$

4) The cuspidal curve $y^2 - x^3 = 0$, with normalization $\mathbb{C}\{x,y\}/f \to \mathbb{C}\{t\}$ given by : $x = t^2, y = t^3$.
$\Omega^1(\log D)$ is generated by the two forms :

$$\omega_0 = \frac{df}{f}, \qquad \omega_1 = \frac{-3ydx + 2xdy}{f}$$

An easy calculation yields

$$y\omega_1 = -3dx + \frac{df}{f}, \text{ hence } Res(\omega_1) = \frac{x}{y} = \frac{1}{t}.$$

We check from this that $\mathscr{R}_D = \frac{1}{t}\mathbb{C}\{t\} \supsetneq \mathscr{O}_{\widetilde{D}}$

Going further.

Among many generalizations and applications I will only discuss here the following :
When D does not satisfy (A), (B) or (C), how to determine $\mathscr{R}_D \supsetneq \mathscr{O}_{\tilde{D}}$ In the case of curves. A detailed answer in terms of the semigroup of multivaluations is given by Delphine Pol.

Semigroup of a curve.

Let $(D,0)=\bigcup_{i=1}^{p} D_i \subset (S,0)$ be a reduced curve, with normalization :

$$\mathscr{O}_D \hookrightarrow \mathscr{O}_{\tilde{D}} \simeq \mathbb{C}\{t_1\}\oplus\cdots\oplus\mathbb{C}\{t_p\}.$$

The value of $g \in Q(\mathscr{O}_D)$, is the p-uple of valuations w.r. to t_j's

$$\mathrm{val(g)} = (\mathrm{val_1(g)},\cdots,\mathrm{val_p(g)}) \in (\mathbb{Z}\cup\{\infty\})^{\mathrm{p}}.$$

Let $\mathrm{val(I)} \subset \mathbb{Z}^{\mathrm{p}}$ be the set of values on non zero divisors in I.

Definition

The semigroup of D is $\Gamma = \mathrm{val}(\mathscr{O}_{\mathrm{D}}) \subset \mathbb{N}^{\mathrm{p}}$.

There is $\underline{\gamma} \in \mathbb{N}^p$ with $\mathrm{val}(\mathscr{C}_{\mathrm{D}}) = (\gamma_1,\cdots,\gamma_{\mathrm{p}}) + \mathbb{N}^{\mathrm{p}}$.
More generally each fractional ideal $I \subset Q(\mathscr{O}_D)$ has a conductor

$$\nu \in \mathbb{Z}^p,\ \mathrm{val(I)} \supset \nu + \mathbb{N}^{\mathrm{p}}$$

A symmetry result.

Theorem (Delgado)

The ring $\mathscr{O}_D$ is Gorenstein iff the semi group Γ has the following property:

$$\forall v \in \mathbb{Z}^p, v \in \Gamma \Longleftrightarrow \Delta(\gamma - v - (1, \cdots, 1), \mathscr{O}_D) = \emptyset$$

For sake of simplicity we limit ourselves to th irreducible case :

$$\forall v \in \mathbb{Z}, v \in \Gamma \Longleftrightarrow \gamma - v - 1 \notin \Gamma$$

This result extends to dual couples like $\mathscr{J}_D$ and $\mathscr{R}_D$.

Theorem (D. Pol)

Let D be a plane branch. Then the set of values of the module of logarithmic residues is determined by $\mathscr{J}_D$*:*

$$v \in \mathrm{val}(\mathscr{R}_{\mathrm{D}}) \Longleftrightarrow \gamma - \mathrm{v} - 1 \notin \mathrm{val}(\mathscr{J}_{\mathrm{D}})$$

Link with Kähler differentials We consider the module Ω^1_C, and its set of values, giving each dt the value 1.

Proposition (G, D.Pol)

We have: $\mathrm{val}(\mathscr{J}_{\mathrm{C}}) = \gamma + \mathrm{val}(\Omega^1_{\mathrm{C}}) - 1$

We use a formula by R. Piene's formula relating these fractionnal ideals with the ramification locus $\mathscr{R}_\nu$ of the normalization.

Corollary (D. Pol)

$$v \in \mathrm{val}(\mathscr{R}_{\mathrm{D}}) \Longleftrightarrow \Delta(-\mathrm{v}, \Omega^1_{\mathrm{D}}) = \emptyset$$

Example Let D be the curve $y^5 - x^6$.

$$\mathrm{val}(\Omega^1_{\mathrm{D}}) = \mathrm{val}(\mathscr{O}_{\mathrm{D}}) \setminus \{0\} = \{5, 6, 10, 11, 12, 15, 16, 17, 18, 20, ...\},$$
$$\mathrm{val}\mathscr{R}_{\mathrm{D}} = \{-19, -14, -13, -9, -8, -7, -4, -3, -2, -1\} \cup \mathbb{N}.$$

Notice that $\dim \mathscr{R}_D / \mathscr{O}_D = \dim \mathscr{O}_D / \mathscr{J}_D = \tau$ the Tjurina number and $\dim \mathscr{R}_D / \mathscr{O}_{\tilde{D}} = \tau - \delta$.

We give a picture for the case of $f = (x^2 - y^3)(x^4 - y^3)$, $\mu = 19$, $\delta = 10$, $\tau = 17$, $\gamma = (8, 12)$, $\gamma_{\mathscr{J}} = (12, 20)$.

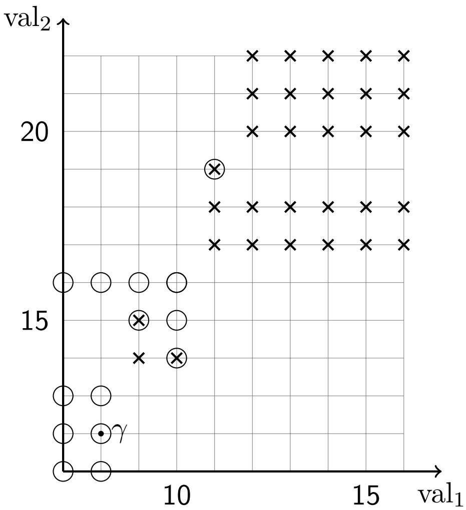

Figure: Semigroup of $\mathscr{J}_D$

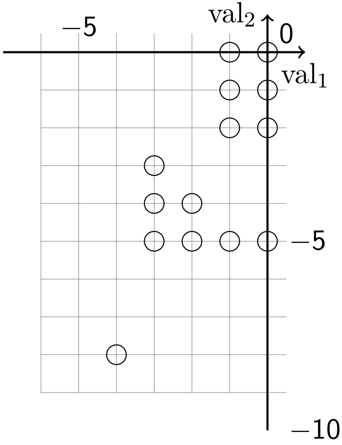

Figure: Multi-values of $\mathscr{R}_D$

In this way $\mathscr{R}_D$ is related to the problem of moduli space for plane branches with a given semi group Γ

Theorem (Hefez,Hernandez)

The set of analytic classes $\mathscr{M}$ of plane branches with given topological type satisfies

$$\mathscr{M} = \bigcup_{\Omega^1_D=\Omega} \mathscr{M}_\Omega.$$

Each $\mathscr{M}_\Omega$ is separated and a quotient by a finite group of an affine open space.

Happy birthday, Tony!

Thank you for your attention