Solving Linear Equations

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Academia Sinica

Jordan Types of Artinian Algebras
and
Geometry of Punctual Hilbert Schemes

A conference to celebrate the 80th birthday of Anthony Iarrobino

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- **Joint work** with I-Hsin Tsai, National Taiwan University.
- **Motivaition:** Our goal is to realize Grothendieck duality concretely by working on residual complexes, which is built up by certain injective modules.
- **Baer's criterion**: A module is injective if and only if certain systems of linear equations are solvable.
- **Main results:** We will construct an injective hull of the residue field of a Noetherian local ring without using Zorn's lemma.
- **Examples:** solving systems of linear equations in an Artinian Gorenstein local ring.

Let κ be a field and $A = \kappa[X]/\langle X^{n+1} \rangle = \kappa + \kappa x + \kappa x^2 + \cdots + \kappa x^n$.

- In the residue field of A, the equation
$$xT = 1$$
does not have solutions.
- Identify the residue field with the module $\langle x^n \rangle$, the above equation is written as
$$xT = x^n,$$
which has a solution in $\langle x^{n-1}, x^n \rangle = \langle x^{n-1} \rangle$.
- The module $\langle x^{n-1} \rangle$ is not "linearly closed" in the sense that some linear equations, for instance, the equation
$$xT = x^{n-1}$$
can not be solved in $\langle x^{n-1} \rangle$. But it can be solved in $\langle x^{n-2} \rangle$.
- For $a_0, \ldots, a_n \in \kappa$, a necessary condition for the equation
$$xT = a_0 + a_1 x + \cdots + a_n x^n$$
to be solvable is $a_0 = 0$. If $a_0 = 0$, the equation is solvable.
- A is an injective hull of its residue field.

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- $\mathfrak{m}^3 = 0$,
- $\mathfrak{m} = \kappa x + \kappa y + \kappa z + \kappa x^2$ is the unique maximal ideal of A.
- $\mathfrak{m}^2 = \langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle \simeq A/\mathfrak{m}$.
- In $\mathfrak{m}^2$, the system

$$\begin{aligned} xT &= ax^2 \quad (a \in \kappa), \\ yT &= by^2 \quad (b \in \kappa), \\ zT &= cz^2 \quad (c \in \kappa) \end{aligned}$$

  of equations has a solution $ax + by + cz$ in $\mathfrak{m}$.
- In $\mathfrak{m}$, the consistent system

$$\begin{aligned} xT &= a_{11}x + a_{12}y + a_{13}z + a_{14}x^2 \quad (a_{1i} \in \kappa), \\ yT &= b_{11}x + b_{12}y + b_{13}z + b_{14}y^2 \quad (b_{1i} \in \kappa), \\ zT &= c_{11}x + c_{12}y + c_{13}z + c_{14}z^2 \quad (c_{1i} \in \kappa) \end{aligned}$$

  of equations has a solution $a_{11} + a_{14}x + b_{14}y + c_{14}z$ in A.
- **Exercise**: Show that every consistent system of equations in A can be solved.

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- Let A be a commutative ring with an identity element. The structure of an A-module M can be understood through system

$$\{a_{i1}T_1 + \cdots + a_{i\ell}T_\ell = \alpha_i\}_{i\in\Gamma}$$

of A-linear equations together with its solutions, where $T_1, T_2, \ldots, T_\ell$ are unknowns, $a_{ij} \in A$, $\alpha_i \in M$ and Γ is an index set.

- A necessary condition for the above system having a solution is

$$b_1a_{i_1j} + \cdots + b_\ell a_{i_\ell j} = 0 \text{ for } 1 \le j \le \ell \implies b_1\alpha_{i_1} + \cdots + b_\ell\alpha_{i_\ell} = 0,$$

where $i_j \in \Gamma$ and $b_i \in A$. Such a condition on the module M is called a *constraint* from the underlying ring A.

- A system of A-linear equations in M is called *consistent*, if it satisfies all constraints from A. A consistent system of linear equations is simply called a *consistent system*.

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- The system

$$\{a_i T = \alpha_i\}_{i \in \Gamma}$$

of equations in T is consistent if and only if $\alpha_i = f(a_i)$ for some A-linear map $f : I \to M$, where I is the ideal generated by a_i.

- Referring to an ideal I of A, we call such a consistent system an I-system in M and identify it with an element in $\mathrm{Hom}_A(I, M)$.
- A solution to an I-system f in M is an element $\alpha \in M$ such that $f(a) = a\alpha$ for all $a \in I$, in other words, f lifts to A.

Baer's criterion

M is an injective module if and only if any I-system in M has a solution.

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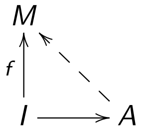

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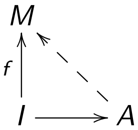

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fields	modules (our terminology)	modules (existing terminology)
algebraic equation	linear equation	linear equation
algebraic extension	linear extension	essential extension
algebraically closed field	linearly closed module	injective module
algebraic closure	linear closure	injective hull

Linear extension

An A-module N containing M is called a *linear extension* of M, if every element of N is a solution to a non-zero I-system in M. For a linear extension N of M, we also say that N is linear over M.

Linear closure

For an ideal I of A, an A-module M is called I-closed if every I-system in M has a solution. The module M is called *linearly closed* if it is I-closed for any ideal I of A. A linear extension of M is called a *linear closure* of M if the extension is linearly closed.

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Initial condition

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. We say that an A-module M satisfies the initial condition if every $\mathfrak{m}$-system in $\mathfrak{m}M$ has a solution in M.

- Example: The residue field of A satisfies the initial condition.

Solution module up to the first order

Let M be an A-module satisfying the initial condition. There exist an A-module $E_1(M)$ and an exact sequence

$$0 \to M \to E_1(M) \to \operatorname{Hom}_A(\mathfrak{m}, M)/\operatorname{Hom}_A(\mathfrak{m}, \mathfrak{m}M) \to 0.$$

- Every $\mathfrak{m}$-system in M has a solution in $E_1(M)$.
- $E_1(M)$ is a linear extension of M.
- If $\omega \in E_1(M)$ is a solution to an $\mathfrak{m}$-system in $\mathfrak{m}M$, then $\omega \in M$.

- If M satisfies the initial condition, so does $E_1(M)$.
- Starting from $E_0(M) := M$, we define

$$E_i(M) := E_1(E_{i-1}(M))$$

for $i \geq 1$ and call $E_i(M)$ the solution module of M up to the i-th order.

- We obtain a filtration

$$0 \subset E_0(M) \subset E_1(M) \subset E_2(M) \subset \cdots$$

Solution module

The direct limit $E(M)$ of $\{E_i(M)\}_{i \geq 0}$ is called the solution module of M.

Theorem

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$, and let M be an A-module satisfying the initial condition. Then $E(M)$ is minimal among $\mathfrak{m}$-closed modules containing M.

Module with zero-dimensional support

An A-module N has zero-dimensional support if every element of N is annihilated by some power of $\mathfrak{m}$.

- The residue field of A has zero-dimensional support.

Theorem

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$, and let M be an A-module that satisfies the initial condition. If M has zero-dimensional support, then $E(M)$ is a maximal linear extension of M.

Theorem

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$, and let M be an A-module that satisfies the initial condition. If M has zero-dimensional support, then $E(M)$ is a linear closure of M.

Corollary

Let M be an A-module with zero-dimensional support. If M is $\mathfrak{m}$-closed, it is linearly closed.

THANK YOU!