Families of symmetric and skew-symmetric matrices and vector bundles

Emilia Mezzetti

UNIVERSITÀ DEGLI STUDI DI TRIESTE

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Linear systems of matrices of constant rank

$V \subset \mathrm{M}_{a,b}(\mathbb{K})$: vector space of $a \times b$ matrices over a field $\mathbb{K}$

Definition

V is a *space of matrices of constant rank r* if all its nonzero elements have the same rank r.

Main questions:

For fixed values of the parameters a, b, r and field $\mathbb{K}$

1. Find max dim V;
2. Construct explicit examples;
3. Classify spaces V.

- Classical work of Kronecker and Weierstrass; also Gantmacher.
- For $\mathbb{K}$ algebraically closed, interesting relation with vector bundles on projective spaces and their invariants, first studied in [J. Sylvester 1986], [Westwick 1987,1990,1996], [Eisenbud-Harris 1988].

Relation with vector bundles

If $\dim V = n+1$, we interpret V as an $a \times b$ matrix M whose entries are linear forms in $n+1$ variables: M defines a morphism

$$\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus b} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^{\oplus a}$$

whose kernel and cokernel are vector bundles on $\mathbb{P}^n$, K of rank $b-r$, E of rank $a-r$:

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus b} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^{\oplus a} \longrightarrow E \longrightarrow 0,$$

Examples (Constant rank 2)

$$\begin{pmatrix} 0 & x & y & z \\ x & y & z & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}$$

Bounds on dimension

From now on $\mathbb{K} = \bar{\mathbb{K}}$ and $\mathrm{char}(\mathbb{K}) = 0$.

Notation

$$\ell(a, b; r) := \max\{\dim V \mid V \subset \mathrm{M}_{a,b}(\mathbb{K}) \textit{ is of constant rank } r\}.$$

[Westwick 1987]: from a computation of invariants and examples

$$b - r + 1 \leq \ell(a, b; r) \leq a + b - 2r + 1, \quad \text{for } 2 \leq r \leq a \leq b.$$

Westwick's bounds are in general not sharp, and the problem is still open.

Remark

If $a = b$, $\quad a - r + 1 \leq \ell(a, a; r) \leq 2(a - r) + 1.$

If moreover $r = a - 2$, $\quad 3 \leq \ell(a, a; r) \leq 5.$

Square (skew-)symmetric matrices

What if the matrices are square and (skew)-symmetric?

Dualizing

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus a} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^{\oplus a} \longrightarrow E \longrightarrow 0,$$

and twisting by -1 we get:

$$0 \longrightarrow E^*(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus a} \xrightarrow{{}^tM} \mathcal{O}_{\mathbb{P}^n}^{\oplus a} \longrightarrow K^*(-1) \longrightarrow 0.$$

From ${}^tM = \pm M$ we obtain an isomorphism $K \simeq E^*(-1)$, and $c_1(E) = \frac{r}{2}$.

In particular, if the (constant) corank is 2, then $K \simeq E(-\frac{a}{2})$.

Skew-symmetric matrices: results

Skew-symmetric matrices

- [Manivel - M. 2005]: classification of the spaces of skew-symmetric matrices of order 6 and constant rank 4 up to the action of the projective linear group PGL_6, and $\ell_{skew}(6,6;4) = 3$.
- [Fania - M. 2011]: classification of all the spaces of dimension 2 up to $SL(a)$-actions; $\ell_{skew}(8,8;6) = 3$ and characterization of the corresponding rank two bundles E on $\mathbb{P}^2$.
- [Boralevi - Faenzi - M. 2013]: existence of spaces of dimension 4 of skew-symmetric matrices of constant corank 2 of order 10 and 14; $\ell_{skew}(10,10;8) = \ell_{skew}(14,14;12) = 4$

Symmetric matrices

Symmetric matrices = linear systems of quadrics

- [Ilic - Landsberg 1999]:

$$\ell_{sym}(r+2, r+2; r) = \begin{cases} 3 & \text{if } r \text{ is even} \\ 1 & \text{if } r \text{ is odd.} \end{cases}$$

- [Boralevi-M 2022]:
  explicit expression for the dimension of every GL_{n+1}-orbit of pencils of symmetric matrices of order $n+1$ of constant non-maximal rank

Skew-symmetric matrices as tensors

- Skew-symmetric matrices of order $n+1$ can be interpreted as skew-symmetric tensors in $\wedge^2\mathbb{K}^{n+1}$ (note change in notation)
- Matrices of rank 2 correspond to the Grassmannian $\mathbb{G}(1,n)$ of lines in $\mathbb{P}^n$
- There is a natural filtration:

$$\mathbb{G}(1,n) \subset \sigma_2(\mathbb{G}(1,n)) \subset \sigma_3(\mathbb{G}(1,n)) \subset \ldots \subset \mathbb{P}(\wedge^2\mathbb{K}^{n+1})$$

  corresponding to skew-symmetric matrices of rank

$$\{\mathrm{rk}=2\} \subset \{\mathrm{rk} \leq 4\} \subset \{\mathrm{rk} \leq 6\} \subset \ldots \subset \{\mathrm{rk} \leq n+1\}.$$

The projectivization of a linear space of skew-symmetric matrices of order $n+1$ and constant rank $2k$ (necessarily even) is a projective space contained in the stratum

$$\sigma_k(\mathbb{G}(1,n)) \setminus \sigma_{k-1}(\mathbb{G}(1,n)).$$

$n = 5$: 6×6 skew-symmetric matrices of constant rank 4

$$\mathbb{G}(1,5) \subset \sigma_2(\mathbb{G}(1,5)) \subset \mathbb{P}(\wedge^2 \mathbb{K}^6) = \mathbb{P}^{14}$$

In the dual space $\check{\mathbb{P}}^{14}$ there is the isomorphic filtration:

$$\mathbb{G}(3,5) \subset \sigma_2(\mathbb{G}(3,5)) \simeq \check{\mathbb{G}}(1,5) \subset \check{\mathbb{P}}^{14}$$

The dual variety $\check{\mathbb{G}}(1,5)$ parametrizing hyperplanes tangent to $\mathbb{G}(1,5)$ is the pfaffian cubic hypersurface of 6×6 skew-symmetric matrices of rank < 6.
$\mathbb{G}(3,5)(\simeq \mathbb{G}(1,5))$ is the singular locus of $\check{\mathbb{G}}(1,5)$: it parameterizes hyperplanes tangent to $\mathbb{G}(1,5)$ at all points representing the lines of a $\mathbb{P}^3$.

We focus on the stratum $\check{\mathbb{G}}(1,5) \setminus \mathbb{G}(3,5)$.

Projective lines in $\check{\mathbb{G}}(1,5) \setminus \mathbb{G}(3,5)$ and vector bundles

[Manivel - M.] There are two orbits of projective lines (i.e. pencils of matrices), under the action of PGL_6 by congruence, corresponding to the two globally generated rank two bundles on $\mathbb{P}^1$ with $c_1 = 2$.

$\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$

$$\left(\begin{array}{cc|cccc} \cdot & \cdot & x & y & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x & y \\ \hline -x & \cdot & \cdot & \cdot & \cdot & \cdot \\ -y & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -x & \cdot & \cdot & \cdot & \cdot \\ \cdot & -y & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

"general lines"

$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$

$$\left(\begin{array}{cc|cccc} \cdot & \cdot & x & y & \cdot & \cdot \\ \cdot & \cdot & \cdot & x & y & \cdot \\ \hline -x & \cdot & \cdot & \cdot & \cdot & \cdot \\ -y & -x & \cdot & \cdot & \cdot & \cdot \\ \cdot & -y & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

"special lines"

The space of these lines is irreducible of dim 22, with an open PGL_6-orbit of general lines and a codim 1 orbit of special lines.

Planes in $\check{\mathbb{G}}(1,5) \setminus \mathbb{G}(1,5)$ and vector bundles

There are four orbits all of dimension 26 of projective planes corresponding to rk 2 globally generated bundles on $\mathbb{P}^2$ with $c_1 = 2$ defining an embedding in $\mathbb{G}(1,5)$

$\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$, $c_2 = 0$

$$\left(\begin{array}{ccccc|c} \cdot & \cdot & \cdot & x & y & \cdot \\ \cdot & \cdot & x & y & z & \cdot \\ \cdot & -x & \cdot & z & \cdot & \cdot \\ -x & -y & -z & \cdot & \cdot & \cdot \\ -y & -z & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

Restricted null correlation bundle, $c_2 = 2$

$$\begin{pmatrix} \cdot & \cdot & \cdot & x & y & z \\ \cdot & \cdot & x & y & \cdot & \cdot \\ \cdot & -x & \cdot & z & \cdot & \cdot \\ -x & -y & -z & \cdot & \cdot & \cdot \\ -y & \cdot & \cdot & \cdot & \cdot & \cdot \\ -z & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$, $c_2 = 1$

$$\left(\begin{array}{ccc|ccc} \cdot & x & y & \cdot & \cdot & \cdot \\ -x & \cdot & z & \cdot & \cdot & \cdot \\ -y & -z & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & x & y \\ \cdot & \cdot & \cdot & -x & \cdot & z \\ \cdot & \cdot & \cdot & -y & -z & \cdot \end{array}\right)$$

Steiner bundle, $c_2 = 3$

$$\left(\begin{array}{cc|cccc} \cdot & \cdot & x & y & z & \cdot \\ \cdot & \cdot & \cdot & x & y & z \\ \hline -x & \cdot & \cdot & \cdot & \cdot & \cdot \\ -y & -x & \cdot & \cdot & \cdot & \cdot \\ -z & -y & \cdot & \cdot & \cdot & \cdot \\ \cdot & -z & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

Quadratic systems and motivation: congruences of lines

There are no $\mathbb{P}^3$s inside $\check{\mathbb{G}}(1,5) \setminus \mathbb{G}(3,5)$.

In [Boralevi-Fania-M. 2021] we focus our attention on **smooth quadric surfaces**. WHY?

A *congruence of lines in* $\mathbb{P}^n$ is a flat family B of lines in $\mathbb{P}^n$ obtained as desingularization of a $(n-1)$-dimensional subvariety B' of the Grassmannian $\mathbb{G}(1,n)$. Consider the incidence correspondence with the natural projection:

$$\Lambda \subset B \times \mathbb{P}^n \xrightarrow{p} \mathbb{P}^n$$

- Order of the congruence is the degree of the map p: number of lines through a general point;
- $P \in \mathbb{P}^n$ is a fundamental point if P belongs to infinitely many lines of B: Φ fundamental locus;
- the schematic image of the ramification divisor of p is F the focal scheme.

Congruences of lines of order 1

1. $\Phi \subset F$,
2. if $l \in B$, then either $l \subset F$ or l contains $n-1$ focal points (counted with multiplicity),
3. B of order > 0: $\Phi = F$ (and codim $F > 1$) **if and only if** order $B = 1$.

Congruences of lines of order 1, and in particular linear congruences of lines, have been studied by Arrondo, Bertolini, Turrini, Ran, De Poi, Peskine,...

Interesting applications/connections to

1. Integrable system (Agofonov - Ferapontov)
2. Degree of irrationality of projective varieties (Bastianelli - Cortini - De Poi, Bastianelli - De Poi - Ein - Lazarsfeld - Ullery)
3. Zak's conjecture on k-normality.

Linear congruences

Linear congruence: $B' = \mathbb{G}(1,n) \cap H_1 \cap H_2 \cap \ldots \cap H_{n-1} = \mathbb{G}(1,n) \cap \Delta$, H_i hyperplanes in $\mathbb{P}(\wedge^2 \mathbb{K}^{n+1}) \rightsquigarrow$ they correspond to points in the dual space $\check{\mathbb{P}}(\wedge^2 \mathbb{K}^{n+1})$, generating a $(n-2)$-space $\check{\Delta}$.

General linear congruences

1. In $\mathbb{P}^3$: lines meeting two fixed lines $\iff$ pencil of hyperplanes in $\mathbb{P}^5 = \mathbb{P}(\wedge^2 \mathbb{K}^6) \iff$ line $\check{\Delta}$ in $\check{\mathbb{P}}^5$ with two points of intersection with $\check{\mathbb{G}}(1,3)$.
2. In $\mathbb{P}^4$: trisecant lines of a projected Veronese surface $\iff$ plane $\check{\Delta}$ in $\check{\mathbb{P}}^9$ disjoint from $\check{\mathbb{G}}(1,4)$.
3. In $\mathbb{P}^5$: 4-secant lines of a degree 7 scroll X over a smooth cubic surface S: the Palatini scroll, which is the focal scheme $\iff$ 3-space $\check{\Delta}$ in $\check{\mathbb{P}}^{14}$, intersecting $\check{\mathbb{G}}(1,5)$ along S, disjoint from $\mathbb{G}(3,5)$.

Special linear congruences

Special positions:

- in $\mathbb{P}^4$ classified by Castelnuovo (1891);
- in $\mathbb{P}^5$ partial results in [De Poi - M., Siena].

To be understood: the geometry of the special cases when $\check{\Delta} \cap \check{\mathbb{G}}(1,5)$ splits as $(\mathit{plane}) \cup (\mathit{quadric})$.

Planes are classified; what about quadrics?

Quadratic systems of skew-symmetric matrices

Q a smooth quadric surface, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and embedded into $\mathbb{P}^3$ through the Segre map. Any line bundle over Q is of the form $\mathcal{O}_Q(a,b) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(b))$, where the π_i's are the projections over $\mathbb{P}^1$.

The existence of a smooth quadric surface $Q \subset \check{\mathbb{G}}(1,5) \setminus \mathbb{G}(3,5)$ implies the existence of an exact sequence of vector bundles on Q:

$$0 \to E(-3,-3) \to \mathcal{O}_Q(-1,-1)^{\oplus 6} \to \mathcal{O}_Q^{\oplus 6} \to E \to 0 \qquad (\star)$$

E is a rank 2 globally generated vector bundle on Q, whose Chern classes satisfy $c_1(E) = (2,2)$ and $0 \le c_2(E) \le 6$.

Rank 2 globally generated vector bundles on Q with $c_1 = (2, 2)$

Globally generated bundles on a smooth quadric surface are classified in [Ballico-Huh-Malaspina 2015].

Decomposable $=$ direct sum of two line bundles
$E = \mathcal{O}_Q(a, b) \oplus \mathcal{O}_Q(2-a, 2-b), \quad 0 \le a, b \le 2.$

(DEC1) $E = \mathcal{O}_Q \oplus \mathcal{O}_Q(2, 2)$, $c_2(E) = 0$;

(DEC2) $E = \mathcal{O}_Q(1, 1)^{\oplus 2}$, $c_2(E) = 2$;

(DEC3) $E = \mathcal{O}_Q(2, 1) \oplus \mathcal{O}_Q(0, 1)$, $c_2(E) = 2$;

(DEC4) $E = \mathcal{O}_Q(2, 0) \oplus \mathcal{O}_Q(0, 2)$, $c_2(E) = 4$.

Indecomposable gg bundles with $c_1 = (2, 2)$ exist if and only if $c_2 = 3, 4, 5, 6, 8$.
Not all of them fit in an exact sequence $(\star)$.

Main result in [Boralevi-Fania-M. 2021]

$\check{\mathbb{G}}(1,5) \subset \check{\mathbb{P}}^{14}$ the cubic Pfaffian hypersurface parametrizing 6×6 skew-symmetric matrices of rank at most 4

Theorem (Existence Theorem)

There exists a smooth quadric surface $Q \subset \check{\mathbb{G}}(1,5)$, *not intersecting the Grassmannian* $\mathbb{G}(3,5)$, *giving rise to an exact sequence*

$$0 \to E(-3,-3) \to \mathcal{O}_Q(-1,-1)^{\oplus 6} \to \mathcal{O}_Q^{\oplus 6} \to E \to 0, \qquad (\star)$$

if and only if *the vector bundle E is either one of the decomposable bundles* (DECi) *or one* (INDj) *of the following list.*

Indecomposable rank 2 vector bundles on Q

(IND1) $0 \to \mathcal{O}_Q \to \mathcal{O}_Q(1,1) \oplus \mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1) \to E \to 0$, $c_2(E) = 3$;

(IND2) $0 \to \mathcal{O}_Q(-1,-1) \to \mathcal{O}_Q^{\oplus 2} \oplus \mathcal{O}_Q(1,1) \to E \to 0$, $c_2(E) = 4$;

(IND3) There is a ses $0 \to \mathcal{O}_Q(1,0) \to E \to \mathcal{I}_Z(1,2) \to 0$, Z is a 0-dim scheme of deg 2. E is stable, $c_2(E) = 4$;

(IND4) There is a ses $0 \to \mathcal{O}_Q \to E \to \mathcal{I}_Z(2,2) \to 0$, Z is a 0-dim scheme of deg 5. E is stable, $c_2(E) = 5$;

(IND5) There is a ses $0 \to \mathcal{O}_Q \to E \to \mathcal{I}_Z(2,2) \to 0$, Z is a 0-dim scheme of deg 6. E is stable, $c_2(E) = 6$.

Main technique: building blocks and projection

- Detect some building blocks of constant rank two corresponding to bundles with $c_1 = (1,1)$;
- construct matrices of constant rank 4 and any size, direct sum of building blocks;
- if the obtained matrices have bigger size, suitably project to a 6×6 matrix while maintaining constant rank.

In terms of vector bundles, try to insert a candidate bundle E in an exact sequence of the form:

$$0 \to \mathcal{O}_Q^{\oplus(\operatorname{rk} F - 2)} \to F \to E \to 0$$

where F is a direct sum of two vector bundles with $c_1 = (1,1)$, and $c_2(F) = c_2(E)$.

The globally generated bundles of any rank on Q with $c_1 = (1,1)$ are:

(i) $\mathcal{O}_Q(1,1)$;

(ii) $\mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1)$;

(iii) $\mathrm{T}\,\mathbb{P}^3(-1)|_Q$;

(iv) $\mathcal{A}_P = \pi_P^*(\mathrm{T}\,\mathbb{P}^2(-1))$, where $\pi_P : Q \to \mathbb{P}^2$ is the projection of center $P \notin Q$.

Each case gives a building block to construct the required matrices.

An example: building block for the decomposable bundle $\mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1)$

It appears in an exact sequence

$$0 \to \mathcal{O}_Q(-2,-1) \oplus \mathcal{O}_Q(-1,-2) \to 4\mathcal{O}_Q(-1) \xrightarrow{M} 4\mathcal{O}_Q \to \mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1) \to 0 :$$

(i) compose the short exact sequence

$$0 \to \mathcal{O}_Q(-1,0) \to 2\mathcal{O}_Q \to \mathcal{O}_Q(1,0) \to 0$$

with itself tensored with $\mathcal{O}_Q(-1)$:

$$\begin{tikzcd} \mathcal{O}_Q(-2,-1) \arrow[r, hook] & 2\mathcal{O}_Q(-1) \arrow[rr] \arrow[dr] & & 2\mathcal{O}_Q \arrow[r, two heads] & \mathcal{O}_Q(1,0). \\ & & \mathcal{O}_Q(-1,0) \arrow[ur] & & \end{tikzcd}$$

(ii) take the direct sum of the sequence obtained with the symmetric one with respect to the rulings.

A corresponding building block is, for instance $M = \begin{pmatrix} \cdot & \cdot & x & y \\ \cdot & \cdot & z & t \\ -x & -z & \cdot & \cdot \\ -y & -t & \cdot & \cdot \end{pmatrix}$: vanishing of Pfaff(M) defines a quadric surface contained in $\check{\mathbb{G}}(1,3)$ as a linear section. It represents a linear congruence of lines in $\mathbb{P}^3$, formed by the lines meeting two skew lines.

Construction of bundles with $c_2 = 4$

A quotient E of the form

$$0 \to \mathcal{O}_Q^{\oplus 2} \to \mathcal{O}_Q(1,0)^{\oplus 2} \oplus \mathcal{O}_Q(0,1)^{\oplus 2} \to E \to 0$$

has $c_2(E) = 4$.
There are three bundles E to be constructed: (DEC4), (IND2) (IND3).
The three cases have different behaviours when restricted to the two rulings of the quadric Q: the decomposable case (DEC4) restricts as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ on both rulings, (IND2) restricts as $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on both rulings, and (IND3) restricts as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ on one ruling and as $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on the other one.

We get the three cases for different choices of the centre of projection.

Details on the projection

The projection

$$\pi_O : \mathbb{P}^n = \mathbb{P}(\mathbb{K}^{n+1}) \to \mathbb{P}^{n-1} = \mathbb{P}(\mathbb{K}^n)$$

from a point O induces another projection

$$\pi_{\Lambda_O} : \mathbb{P}(\Lambda^2\mathbb{K}^{n+1}) \to \mathbb{P}(\Lambda^2\mathbb{K}^n)$$

whose centre is $\Lambda_O \subseteq \mathbb{G}(1, n)$: union of the lines through O.

Consider S, a surface contained in $\sigma_r(\mathbb{G}(1, n)) \setminus \sigma_{r-1}(\mathbb{G}(1, n))$, points in S are of the form $\omega = [v_1 \wedge w_1 + \cdots + v_r \wedge w_r]$: v_i, w_i are $2r$ linearly independent vectors; the corresponding points generate a subspace $L_\omega \subset \mathbb{P}^n$ of dimension $2r - 1$.

The matrices of $\pi_{\Lambda_O}(S)$ have constant rank $2r$ if and only if the point O does not belong to the union of the spaces L_ω, as ω varies in S.

The bundles with $c_2 = 4$

We have to choose a line $L \subset \mathbb{P}^7$, centre of projection.

$F = \mathcal{O}_Q(1,0)^{\oplus 2} \oplus \mathcal{O}_Q(0,1)^{\oplus 2}$ defines a map $\psi : Q \to \mathbb{G}(3,7)$: each direct summand defines $\pi_i : Q \to \mathbb{P}^1$, identify the codomains with general lines $l_i \subset \mathbb{P}^7$, then ψ maps a point $P \in Q$ to the 3-space generated by the images $\pi_i(P)$.

We get different bundles E according to the position of L with respect to the 5-spaces S_i, dual of the lines l_i.

The most special situation is when L meets all the S_i: E splits and we get case (DEC4).
If L meets two of the S_i E has different splitting type on the rulings of Q, we get case (IND3).
The general case (IND2) is obtained when L is disjoint from all the spaces S_i.

For other positions of L the rank does not remain constant by projecting.

Application: a table

Assume that $\check{\Delta} \cap \check{\mathbb{G}}(1,5) = \Pi \cup Q$, Π a plane, Q a smooth quadric, with Π, Q of constant rank 4.
Π corresponds to a bundle F, Q corresponds to a bundle E. The previous results give the existence of the following:

E **on** Q	$c_2(E)$	F **on** $\mathbb{P}^2$	$c_2(F)$
$\mathcal{O}_Q(2,1) \oplus \mathcal{O}_Q(0,1)$	2	Steiner	3
$\mathcal{O}_Q(2,0) \oplus \mathcal{O}_Q(0,2)$	4	$\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$	1
(IND1)	3	Null corr. of $\mathbb{P}^3$ restricted to $\mathbb{P}^2$	2
(IND2)	4	$\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$	1
(IND3)	4	$\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$	1
(IND4)	5	$\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$	0

The focal locus splits in two components, of degrees $8 - c_2(E)$ and $4 - c_2(F)$, whose union has degree 7, so $c_2(E) + c_2(F) = 5$.
In the other examples we found that $\langle Q \rangle \subset \check{\mathbb{G}}(1,5)$.

Thank you for your attention ... and congratulations to Tony!