Maximal Hilbert functions of Artinian quotients of a product ring.

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Salute

- First, congratulations to Tony with his 80 years.

  You have been a prominant mathematician of the mathematical community, starting with works on Hilbert schemes at MIT and you have many important publications in good journals, starting with :

  “Reducibility of the Families of 0-Dimensional Schemes on a Variety..” Inventiones mathematicae 15 (1971/72): 72-77 , and

  “Punctial Hilbert schemes” Bull. Amer. Math. Soc. Vol. 78, No 5, Sept. 1972

- Even though we have not published papers together, we have shared common interests in Hilbert schemes.

- Your Springer Lecture Note with Kanev (SLN 1721) has been an inspiration for me. When I realized that the parameter scheme $\mathbb{P}$Gor(h) you studied had to be smooth for Artinian quotients of a 3-dim. poly. ring; it resulted in a paper: [PGor98] The smoothness and the dimension of $\mathbb{P}$Gor(h) and of other strata of punctual Hilb. scheme J. Algebra, 200(1998)
- In 2001 [in Journal of Algebra 241, "Hilbert functions and Level algebras"], you and Young-Hyun Cho initiated the study of (relatively) compressed quotients C of a proper quotient A of a polynomial ring; these are the C whose length is maximal among quotients of fixed "permissible" socle type $\boldsymbol{t}(q)$. You two gave examples of A where this maximal length is not the a priori bound $\boldsymbol{h}_{\mathfrak{s}}^{\mathrm{I}}(p)$ computed from the socle type and the Hilbert function $\boldsymbol{a}(p)$ of A. A few years later, you and others gave additional examples. In my talk I will sort of generalize your examples looking at quotients of a "product ring" A.

- Another inspiration is your important paper from 1984 in Transactions AMS, generalizing cowork with J. Emsalem

  [Iar84]: Compressed algebras: Artin algebras having given socle degrees and maximal length. Trans. AMS 285(1984)

  Steve Kleiman and I have for several years worked together, at your suggestion and encouragement, on problems naturally arising from your and others work in

  [KK25]: Kleiman, Kleppe. Macaulay Duality and Its Geometry. In: Albano, A., et al. Perspectives on Four Decades of Algebraic Geometry, Volume 1. Progress in Mathematics, vol 351. Birkhäuser, Cham. (2025)

  I'll first recall some results from [KK25].

Macaulay duality

- Now k is a **Noetherian** ring. Given any k-module C, define its *k-dual* by
$$C^* := \mathrm{Hom}_k(C,\, k)\ .$$
- A is **any** fin.gen. graded k-algebra (poss. non-standard) with A_p locally free of rank $\boldsymbol{a}(p)$, $\boldsymbol{a}(0) = 1$, $\boldsymbol{a}(p) = 0$ for $p < 0$.

  Set $(A^\dagger)_q := (A_{-q})^*$, $A^\dagger := \bigoplus_q (A^\dagger)_q$, the *graded dual* of A.
$$(A^\dagger)_q \text{ is locally free of rank } \boldsymbol{a}(-q) =: \boldsymbol{a}^*(q).$$
- Let $\mathbb{F}_A$ (resp.$\mathbb{G}_A$):= the cat. of filtered (resp. graded) A-modules. Call an A-module C *k-Artinian* if C is locally free of finite rank over k. Let $\mathbb{AF}_A \subset \mathbb{F}_A$, $\mathbb{AG}_A \subset \mathbb{G}_A$ and $\mathbb{AM}_A \subset \mathbb{M}_A$ (:= cat. of A-modules) be full subcat. of such C.

Set-up of apolarity

- For A-submodules I of A, and D of $A^\dagger$, let

$$Ann(D) := (0 :_A D) = \{\, \psi \in A \mid \psi \cdot f = 0 \text{ for all } f \in D \,\}.$$

$$(0 :_{A^\dagger} I) := \{\, f \in A^\dagger \mid \psi \cdot f = 0 \text{ for all } \psi \in I \,\}.$$

Example 1 Let $A = P = k[x, y, z]$ and $A^\dagger = k[x^{-1}, y^{-1}, z^{-1}]$, Note that multiplication by x, y or z in $A^\dagger$ correspond to applying the partiel derivatives $\partial/\partial X, \partial/\partial Y, \partial/\partial Z$ without coefficient onto $k[X, Y, Z]$, $X := x^{-1}, Y := y^{-1}, Z := z^{-1}$ Let $f = x^{-1}y^{-2}z^{-2}$. Then the deg. of f is $deg(f) = -5$ and

$$xf = y^{-2}z^{-2}\ , \quad x^2 f = 0\ ,\ yf = x^{-1}y^{-1}z^{-2}\ , \quad y^3 f = 0\ ,\ \text{etc}$$

Thus $\qquad Ann(f) = (x^2, y^3, z^3) \quad , \quad$ and

$C := R/Ann(f)$ has Hilbert function $\boldsymbol{h}_C = (1, 3, 5, 5, 3, 1)$.

- Denote by Ψ_A (resp. $\Delta_{A^\dagger}$) the set of A-submodules $I \subset A$ (resp. $D \subset A^\dagger$) with $A/I \in \mathbb{AM}_A$ (resp. $D \in \mathbb{AM}_A$ and with $A^\dagger/D$ k-flat).
- Let $\boldsymbol{h}^*(q) := \boldsymbol{h}(-q)$. Denote by $\boldsymbol{F}\Psi_A^h \subset \Psi_A$ the subset of all I with $A/I \in \mathbb{AF}_A^h$, and by $\boldsymbol{F}\Delta_{A^\dagger}^{h^*} \subset \Delta_{A^\dagger}$ the subset of all D with $D \in \mathbb{AF}_A^{h^*}$ and with $G_\bullet(A^\dagger/D)$ k-flat (so $A^\dagger/D$ is k-flat)
- Denote by $\boldsymbol{H}\Psi_A^h \subset \boldsymbol{F}\Psi_A^h$ and $\boldsymbol{H}\Delta_{A^\dagger}^{h^*} \subset \boldsymbol{F}\Delta_{A^\dagger}^{h^*}$ the subset of homogeneous I and D.
- If K is a k-algebra and $A_K := A \otimes_k K$ then
$$\boldsymbol{H}\Psi_{A_K}^h \;,\quad \boldsymbol{F}\Psi_{A_K}^h \;,\quad \boldsymbol{H}\Delta_{A_K^\dagger}^{h^*} \;,\quad \boldsymbol{F}\Delta_{A_K^\dagger}^{h^*}$$
are as above with A_K for A. These definitions extend to noetherian schemes $T/Spec(k)$, and by [KK25,(6.2)]:

Theorem (Thm1: Representability)

These definitions give functors in T which are representable by subschemes, say $\mathbb{H}\Psi_A^{\boldsymbol{h}}$, $\mathbb{F}\Psi_A^{\boldsymbol{h}}$, and $\mathbb{H}\Delta_{A^\dagger}^{\boldsymbol{h}^*}$, $\mathbb{F}\Delta_{A^\dagger}^{\boldsymbol{h}^*}$ of a certain $\mathbb{Q}$uot-scheme.

Theorem (Thm2: Macaulay Duality)

Keep the setup above .
(1) Then $I \mapsto (0 :_{A^\dagger} I)$ *gives a bijection* $\Psi_A \cong \Delta_{A^\dagger}$, *with inverse* $D \mapsto (0 :_A D)$. *Also,* $(0 :_{A^\dagger} I) = (A/I)^*$ *and* $A/(0 :_A D) = D^*$.
Further, if I *and* D *correspond, then* D *is a dualizing (or canonical) module for* $\mathbb{AM}_{A/I}$;

(2) The bijection in (1) induces a second bijection, $\boldsymbol{F}\Psi_A^h \cong \boldsymbol{F}\Delta_{A^\dagger}^{h^*}$ *which restricts to a third,* $\boldsymbol{H}\Psi_A^h \cong \boldsymbol{H}\Delta_{A^\dagger}^{h^*}$. *These two bijections commute with taking associated graded modules.*
Thus, Macaulay Duality gives canonical isomorphisms

$$\mathbb{F}\Psi_A^{\boldsymbol{h}} = \mathbb{F}\Delta_{A^\dagger}^{\boldsymbol{h}^*} \quad \text{and} \quad \mathbb{H}\Psi_A^{\boldsymbol{h}} = \mathbb{H}\Delta_{A^\dagger}^{\boldsymbol{h}^*}$$

Proof See [KK25, (3.5)]

Let $\boldsymbol{h}, \boldsymbol{t} : \mathbb{Z} \to \mathbb{Z}$ be non-negative finite functions, $\boldsymbol{h}, \boldsymbol{t} \neq 0$. Set

$$s := s(\boldsymbol{h}) := \sup\{ n \mid \boldsymbol{h}(n) \neq 0 \} \ .$$

Let $C \in \mathbb{AF}_A^{\boldsymbol{h}}$. Then s is **the socle degree** of C , and

Definition 1 The *k-socle* of C and its induced filtration are:

$$\mathrm{Soc}_k(C) := \mathrm{Hom}_A(k, C) = \{ c \in C \mid (F^1 A) \cdot c = 0 \}$$
$$F^n(\mathrm{Soc}_k(C)) := F^n \mathrm{Hom}_A(k,\ C) \quad \text{for all } n.$$

Lemma (Lem1: Socle)

In $\mathbb{F}_A$, there's a canonical isomorphism: $\mathrm{Soc}_k(C) = (C^* \otimes_A k)^*$.

Proof [Iar84, Lem. 2.1] for k a field, or [KK25,(4.3)] which yields

$$\mathrm{Hom}_A(k,\ C) = \mathrm{Hom}_A(C^*,\ k^*) = (C^* \otimes_A k)^* \ \text{ in } \mathbb{F}_A. \quad \square$$

Thus if $C^* \otimes_A k$ is k-Artinian, then so is $\mathrm{Soc}_k(C)$, and their Hilbert functions, say $\boldsymbol{t}^*$ and $\boldsymbol{t}$, satisfy $\boldsymbol{t}^*(p) = \boldsymbol{t}(-p)$.

$\boldsymbol{t}$ is called the k-**socle type** of C and $\boldsymbol{t}^*$ the **generator type** of C^*.

Let $\quad \bar{s} := \inf\{\, n \mid \boldsymbol{t}(n) \neq 0 \,\}$ and let

$$\boldsymbol{g}_{\bar{s}}(p) = \textstyle\sum_{q=\bar{s}}^{s} \boldsymbol{t}(q)\, \boldsymbol{a}(q-p) \text{ and } \boldsymbol{h}_{\bar{s}}^{\mathrm{I}}(p) = \min\{\, \boldsymbol{g}_{\bar{s}}(p),\ \boldsymbol{a}(p) \,\}.$$

(We define g_m and $\boldsymbol{h}_m^{\mathrm{I}}$ more generally below.)

DEF.2 With C of soc.deg. s, set $D := C^*$ and

$$\Delta^m D := A(\oplus_{j=m}^{s} D_{-j}) \subset D$$

Fix $\boldsymbol{h}_m$. If $\Delta^m D \in \boldsymbol{H}\Delta_D^{\boldsymbol{h}_m^*}$ for all m, denote by $\boldsymbol{H}\Lambda_A^{\{\boldsymbol{h}_m\}}$ the set of corresponding C. It extends to a representable functor. Let

$$\boldsymbol{g}_m(p) := \textstyle\sum_{q=m}^{s} \boldsymbol{t}(q)\, \boldsymbol{a}(q-p), \qquad (1)$$

$$\boldsymbol{h}_m^{\mathrm{I}}(p) := \min\{\, \boldsymbol{g}_m(p),\ \boldsymbol{a}(p) \,\}. \qquad (2)$$

Remark 1 (i) If $\boldsymbol{t}^*$ is the generator type of C^*, there's a map

$$N := \oplus_{q\in\mathbb{Z}} A(q)^{\oplus \boldsymbol{t}(q)} \twoheadrightarrow C^* \ .$$

So $g_{\bar{s}}(p)$ is the rank of N_{-p}. So $\boldsymbol{h}(p) \le \boldsymbol{h}^{\mathrm{I}}_{\bar{s}}(p)$ as rank $C^*_{-p} \le \boldsymbol{a}(p)$

(ii) Also $\boldsymbol{g}_0(p) = \boldsymbol{g}_{\bar{s}}(p)$ for $p \ge 0$ as $g_m(p) =$ rank $\oplus_{q=m}^{s} A(q)_{-p}^{\oplus \boldsymbol{t}(q)}$
(iii) As $\Delta^{\bar{s}} C^* = C^*$, $\boldsymbol{h}^{\mathrm{I}}_{\bar{s}}$ is the Hilbert function of a $C \in \boldsymbol{H}\Lambda_A^{\{\boldsymbol{h}^{\mathrm{I}}_m\}}$

Proposition (Prop1: Maximality of $\boldsymbol{h}^{\mathrm{I}}_m$, [KK25, (8.2)])

Fix $C \in \boldsymbol{H}\Lambda_A^{\{\boldsymbol{h}_m\}}$ of k-socle type $\boldsymbol{t}$. Then $\boldsymbol{h}_m(p) \le \boldsymbol{h}^{\mathrm{I}}_m(p)\ \forall\ m,\ p$.

Definition 3 We say $C \in \boldsymbol{H}\Lambda_A^{\{\boldsymbol{h}_m\}}$ of k-socle type $\boldsymbol{t}$ is I-compressed if $\boldsymbol{h}_m = \boldsymbol{h}^{\mathrm{I}}_m$ for any m. Note the Hilb. funct. of $(\Delta^m C^*)^* = \boldsymbol{h}_m$.

Remark 2. $C \in \boldsymbol{H}\Psi_A^{\boldsymbol{h}}$ of socle type $\boldsymbol{t}$ is compressed iff $\boldsymbol{h} = \boldsymbol{h}^{\mathrm{I}}_{\bar{s}}$ by [Iar84]

For the latter Def. and Rem. to hold, we restrict to permissible $\boldsymbol{t}$:

Permissible socle types Let

$$v_m := \inf\{\, p \mid \boldsymbol{a}(p) > \boldsymbol{g}_m(p) \,\}$$

and define $b_1 := b_1(A)$ by

$$b_1 := v_0 \qquad \text{if } \boldsymbol{a}(v_0 - 1) < \boldsymbol{g}_0(v_0 - 1), \text{ and}$$
$$b_1 := v_0 - 1 \text{ if } \boldsymbol{a}(v_0 - 1) = \boldsymbol{g}_0(v_0 - 1)$$

Then call $\boldsymbol{t}$ *permissible* (for A), see [KK25,(8.4)(2)], if

$$\bar{s} \geq b_1 \quad \textit{and} \quad \boldsymbol{a}(p) > \boldsymbol{g}_m(p) \textit{ for } v_m \leq p \leq s \textit{ and all } m.$$

Remark 3 From now on k is a field.Then C is compressed iff

(*) $\dim C_p = \boldsymbol{a}(p)$ for $p < v_0$ and $\dim C_p = \boldsymbol{g}_{\bar{s}}(p)$ for $p \geq v_0$, **or** (we can in **(*)** replace $p \geq v_0$ by $p \geq b_1$).

As there are surjections $A \xrightarrow{\eta} C$ and $N := \bigoplus_{q \in \mathbb{Z}} A(q)^{\oplus \boldsymbol{t}(q)} \xrightarrow{\epsilon} C^*$ and as $\dim N_{-p} = \boldsymbol{g}_{\bar{s}}(p)$ and $\dim A_p = \boldsymbol{a}(p)$, **(*)** is equiv. to:
(**) ϵ (resp. η) is an isomorph. in deg. $-p \leq -b_1$ (resp. $p < v_0$).

Generalizing Cho-Tony's examples (C-TEx) in J.Alg 2001

They study level artinian quotients C of $A = R/((xy, xz) + L)$, $R = k[x, y, z]$ a poly. ring and L an ideal of $k[y, z]$ such that $\mathrm{Proj}(A) \subset \mathbb{P}^2$ is a punctual scheme. They show that there's no level (i.e. $\bar{s} = s$) C with Hilbert function $\boldsymbol{h}_s^{\mathrm{I}} = (1, 3, 4, 5, 6, 2)$ of socle deg. 5; and in fact no level C with

$$\boldsymbol{h}_s^{\mathrm{I}} = (1, 3, 4, 5, 6, ...6, 6, 2)$$

for any soc. deg. ≥ 5, but there are artinian C with

$$\boldsymbol{h}_C = (1, 3, 4, 5, 6, ...6, 5, 2)$$

(e.g. take $A = R/(xy, xz, z^5)$) where $\boldsymbol{h}_C$ is the maximal one

Note the ring A is of the form $(A^1 \otimes_k A^2)/m_1 m_2$ where

$$A^1 = k[x],\ A^2 = k[y, z]/(z^5) \text{ with } m_1 = (x),\ m_2 = (y, z)$$

(their Ex. cover also: $\mathrm{Proj}(A^2)$ consists of 5 smooth points on the line $\mathrm{Proj}(k[y, z])$; they give several other examples too)

Example 2 [ChoTony]
Let $A = k[x,y,z]/(xy,xz,z^5)$, and $C^* = A(f_1,f_2)$ with

$$f_1 = x^{-s} + y^{-s+2}z^{-2} \quad , \quad f_2 = y^{-s+4}z^{-4} \ \text{ for } s \geq 5.$$

Then $A_1.(f_1,f_2)$ is 5-dimensional and $x.f_2 = 0$ is a lin. relation!
Indeed

$$\begin{aligned} xf_1 &= x^{-s+1} \quad , & xf_2 &= 0 \\ yf_1 &= y^{-s+3}z^{-2} \quad , & yf_2 &= y^{-s+5}z^{-4} \\ zf_1 &= y^{-s+2}z^{-1} \quad , & f_2 &= y^{-s+4}z^{-3} \end{aligned}$$

So $\boldsymbol{h}_C = (1,2,3,....,5,2)$ and;

$$\to A(s-1) \xrightarrow{[0,x]^{tr}} A(s)^2 \twoheadrightarrow C^* \ \text{ is exact in deg}. \ \leq -s+1$$

Set-up for "Product ring" $A = (A^1 \otimes_k A^2)/m_1 m_2$,with $A^1 = P/I$ $A^2 = Q/J$ where $I \subset P$, $J \subset Q$ are graded ideals in two poly. rings $P = k[x_1, ..., x_n]$, $Q = k[y_1, .., y_m]$ with irrel. maximal ideals m_i. So

$$A = k[x_1, .., x_n, y_1, .., y_m]/(m_1 m_2 + RI + RJ) \; with \; R = P \otimes_k Q$$

Note $A_p = A^1_p \oplus A^2_p$, $p > 0$ and $A^1_0 = A^2_0 = k$ with k a field.

Let C_0 be a graded (and C a poss. non-graded) quotient of A, and suppose there's a set of homogen. generators $\{f_1 + g_1, ..., f_\tau + g_\tau\}$ of C_0^* of degree $e_1 \leq e_2 \leq ..., e_\tau < 0$ (so $e_i = deg \; f_i = deg \; g_i$) with $e_1 = -s$, and a function $\boldsymbol{t}$ such that

$$N := \bigoplus_{q \in \mathbb{Z}} A(q)^{\oplus \boldsymbol{t}(q)} \; = \; \bigoplus_{i=1}^{\tau} A(-e_i) \twoheadrightarrow C_0^*.$$

Let $C_1^* := A^1(f_1, ..., f_\tau)$ and $C_2^* := A^2(g_1, ..., g_\tau)$. Then C_1 and C_2 are quotients of A^1 and A^2, say of socle types $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ necessarily satisfying $\boldsymbol{t}_1 \leq \boldsymbol{t}$ and $\boldsymbol{t}_2 \leq \boldsymbol{t}$. Suppose $\boldsymbol{t}_2 = \boldsymbol{t}$, and that $\{g_1, ..., g_\tau\}$ is a min. set of homogen. generators of C_2^*. Moreover suppose the socle types $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are permissible for A^1 and A^2, and that C_1 and C_2 are compressed for their socle types $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$.

Then it's easy to see that $\boldsymbol{t}$ is the socle type of C_0.

Proposition (Prop2: Maximality of $\boldsymbol{h}_{C_0}$)

With C_i^* as above, the Hilbert function $\boldsymbol{h}_{C_0}$ is given by

$$\boldsymbol{h}_{C_0}(p) = \boldsymbol{h}_{C_1}(p) + \boldsymbol{h}_{C_2}(p) - \boldsymbol{t}_1(p) \ \text{ for } p > 0 \ \text{ where } \boldsymbol{h}_{C_i} = \boldsymbol{h}^{\mathrm{I}}_{\bar{s}_i, A^i} \quad (3)$$

Moreover any quotient C_0' of A of socle type $\boldsymbol{t}$ satisfy;

$$\boldsymbol{h}_{C_0'}(p) \leq \boldsymbol{h}_{C_0}(p) \quad \text{for all } p \quad (4)$$

provided either $\boldsymbol{t}_1 = \boldsymbol{t}$ **or** *$\boldsymbol{h}_{C_1}(p) = \boldsymbol{a}^1(p) := \dim A^1(p)$, for $p < s$.*

Proof To prove (3), we compare C_0^* with $C_1^* + C_2^*$. For $q = 1, 2$ let $a_i^q \in A_i^q$ be arbitrary and recall $A_0 = A_0^q = k$. Then

$$C_1^* + C_2^* = A^1(f_1, .., f_\tau) + A^2(g_1, .., g_\tau) = A(f_1, .., f_\tau, g_1, .., g_\tau) \quad (5)$$

as $a_i^2 f_j = 0 = a_i^1 g_j$ for $i > 0$, e.g. $y_i f_j = 0 = x_i g_j$ For the same reason

$$A_i(f_j, g_j) = A_i(f_j + g_j), \quad \textit{for } i > 0 \ \textit{ and any } j \qquad (6)$$

as $(a_i^1 + a_i^2) f_j = a_i^1 f_j = a_i^1 (f_j + g_j)$ and ditto for $a_i^2 g_j$.

Thus, as $C_0^* = A(f_1 + g_1, .., f_\tau + g_\tau)$, the leftmost vertical arrow in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (A_1 C_0^*)_q & \longrightarrow & (C_0^*)_q & \longrightarrow & (C_0^*/(A_1 C_0^*))_q & \longrightarrow & 0 \\ & & \downarrow & \circ & \downarrow & \circ & \downarrow & & \\ 0 & \rightarrow & (A_1(C_1^* + C_2^*))_q & \rightarrow & (C_1^* + C_2^*)_q & \rightarrow & ((C_1^* + C_2^*)/(A_1(C_1^* + C_2^*)))_q & \rightarrow & 0 \end{array}$$

is an equality. Moreover the middle vertical arrow yields obviously an injective map into $(C_1^* + C_2^*)_q$ where the $" + "$ is a direct sum for $q < 0$ as $(A^\dagger)_q = (A^1)_q^\dagger \oplus (A^2)_q^\dagger$. As the rightmost downarrow yields a map between the dual of the socles of C_0 and $C_1 + C_2$ by Lemma 1, we get, with $p = -q > 0$:

$$\boldsymbol{h}_{C_1}(p) + \boldsymbol{h}_{C_2}(p) - \boldsymbol{h}_{C_0}(p) = \boldsymbol{t}_1(p) + \boldsymbol{t}(p) - \boldsymbol{t}(p)$$

which yields (3), as desired.

To prove (4), take any $C_0'^* = A(f_1' + g_1', .., f_\tau' + g_\tau')$ of socle type $\boldsymbol{t}$ and define $C_1'^* = A(f_1', .., f_\tau')$ and $C_2'^* = A(g_1', .., g_\tau')$, their socle types are $\boldsymbol{t}_1'$ and $\boldsymbol{t}'$, say, for which we at least know $\boldsymbol{t}_1' \le \boldsymbol{t}$ and $\boldsymbol{t}' \le \boldsymbol{t}$. As we no place above used that the Hilbert functions of C_1 and C_2 are maximal, the arguments above yield

$$\boldsymbol{h}_{C_0'}(p) = \boldsymbol{h}_{C_1'}(p) + \boldsymbol{h}_{C_2'}(p) - \boldsymbol{t}_1'(p) \quad \textit{for } p > 0 \tag{7}$$

First note that (4) holds for $p = s$ (resp. $p = 0$) as their Hilbert functions for $p = s$ coincide with their socle types for $p = s$ (resp. as $\boldsymbol{t}_1$ is permissible). So let's us suppose $0 < p < s$ below.

Let $\boldsymbol{H}_{C_1'}(p) := \boldsymbol{h}_{C_1'}(p) - \boldsymbol{t}_1'(p)$, $\boldsymbol{H}_{C_1}(p) := \boldsymbol{h}_{C_1}(p) - \boldsymbol{t}_1(p)$ and assume $\boldsymbol{t}_1(p) = 0$. Then by (3) and (7), we get;

$$\boldsymbol{h}_{C_0'}(p) = \boldsymbol{H}_{C_1'}(p) + \boldsymbol{h}_{C_2'}(p) \le \boldsymbol{H}_{C_1}(p) + \boldsymbol{h}_{C_2}(p) = \boldsymbol{h}_{C_0}(p)\ , \tag{8}$$

and hence we get (4), provided we can show

$$\boldsymbol{h}_{C_2'}(p) \le \boldsymbol{h}_{C_2}(p) \quad \textit{and} \quad \boldsymbol{H}_{C_1'}(p) \le \boldsymbol{H}_{C_1}(p) \tag{9}$$

To see $\boldsymbol{h}_{C_2'}(p) \le \boldsymbol{h}_{C_2}(p)$, recall $\boldsymbol{t}' \le \boldsymbol{t}$ where $\boldsymbol{t}$ is the socle type of the A^2-quotient C_2. Using (1) and (2) for $m = p$, first with $\boldsymbol{a}^2, \boldsymbol{t}_1$ for $\boldsymbol{a}, \boldsymbol{t}$ and next with $\boldsymbol{a}^2, \boldsymbol{t}$ for $\boldsymbol{a}, \boldsymbol{t}$, we get the first part of (9).

To see

$$\boldsymbol{H}_{C_1'}(p) \le \boldsymbol{H}_{C_1}(p) := \boldsymbol{h}_{C_1}(p) - \boldsymbol{t}_1(p) \qquad (10)$$

for the A^1-quotient C_1' when $\boldsymbol{t}_1 = \boldsymbol{t}$ we argue exactly as above with $\boldsymbol{a}^1$ for $\boldsymbol{a}^2$, and we get $\boldsymbol{h}_{C_1'}(p) \le \boldsymbol{h}_{C_1}(p)$. Thus (10) holds when $\boldsymbol{t}_1(p) = 0$ as $\boldsymbol{H}_{C_1'}(p) \le \boldsymbol{h}_{C_1'}(p)$.

Suppose $\boldsymbol{t}_1(p) \ne 0$. Then $\boldsymbol{a}^1(p) \ge \boldsymbol{g}^1_{\bar{s}_1}(p)$ as $\boldsymbol{t}_1$ is permissible. So by (2), (1), $\boldsymbol{h}_{C_1}(p) = \boldsymbol{g}^1_{\bar{s}_1}(p) := \sum_{q=p}^{s} \boldsymbol{t}(q)\, \boldsymbol{a}^1(q-p)$. As $\boldsymbol{t}_1' \le \boldsymbol{t}$,

$$\boldsymbol{H}_{C_1}(p) := \sum_{q=p+1}^{s} \boldsymbol{t}(q)\, \boldsymbol{a}^1(q-p) \ge \sum_{q=p}^{s} \boldsymbol{t}_1'(q)\, \boldsymbol{a}^1(q-p) - \boldsymbol{t}_1'(p) \ge \boldsymbol{H}_{C_1'}(p)$$

where the last inequality follows from (1), (2) and the maximality of $\boldsymbol{h}^{\mathrm{I}}_m$. Also if $\boldsymbol{h}_{C_1}(p) = \boldsymbol{a}^1(p)$ for $p < s$, we may have $\boldsymbol{t}_1(p) \ne 0$ in which case $\boldsymbol{a}^1(p) = \boldsymbol{g}^1_{\bar{s}_1}(p)$, and the proof above applies to get (9).

Finally if $\boldsymbol{h}_{C_1}(p) = \boldsymbol{a}^1(p)$ for any fixed $p < s$, we have either $\boldsymbol{t}_1(p) = 0$ or $\boldsymbol{g}^1_{\bar{s}_1}(p) = \boldsymbol{a}^1(p)$ as $\boldsymbol{t}_1$ is permissible. In the first case just insert $\boldsymbol{a}^1(p)$ for $\boldsymbol{H}_{C_1}(p)$ in (8) and note that $\boldsymbol{H}_{C_1'}(p) \leq \boldsymbol{a}^1(p)$ as C_1' is a quotient of A^1 (also $\boldsymbol{h}_{C_2'}(p) \leq \boldsymbol{h}_{C_2}(p)$ holds by the proof after (4). As the second case was already treated in the last paragraph above, then (4) is proved. □

Corollary 1 With C_2^*, C_1^* and C_0^* as in Prop.2, suppose $\boldsymbol{t}_1 = \boldsymbol{t}$ and moreover that $\boldsymbol{a}^q(i) := \dim A_i^q$ satisfy

$$\boldsymbol{a}^1(i) = \boldsymbol{a}^2(i) \quad \textit{for all} \quad p \leq s \ .$$

Then $C_0 \in \boldsymbol{H}\Psi_A^{\boldsymbol{h}^{\mathrm{I}}_{\bar{s}}}$.

Proof Note $\bar{s}_i = \bar{s}$ for $i = 1, 2$ as A_i and A have the same $\boldsymbol{t}$. Also $\boldsymbol{a}(i) = 2\boldsymbol{a}^1(i)$ *for* $0 < p \leq s$. Assume $\boldsymbol{t}(p) = 0$. Then just multiply

$$\boldsymbol{g}^1_{\bar{s}}(p) := \textstyle\sum_{q=\bar{s}}^{s} \boldsymbol{t}(q)\, \boldsymbol{a}^1(q-p),\ \boldsymbol{h}^{\mathrm{I}}_{A_1}(p) := \min\{\, \boldsymbol{g}^1_{\bar{s}}(p),\ \boldsymbol{a}^1(p)\,\}$$

by 2 to get $\boldsymbol{h}^{\mathrm{I}}_A = 2\boldsymbol{h}^{\mathrm{I}}_{A^1}$, whence by Prop., $\boldsymbol{h}_{C_0} = 2\boldsymbol{h}_{C_1} = \boldsymbol{h}^{\mathrm{I}}_{\bar{s},A}$. If $\boldsymbol{t}(p) \neq 0$, then $\boldsymbol{a}^1(p) \geq \boldsymbol{g}^1_{\bar{s}}(p)$ and $\boldsymbol{g}_{\bar{s}}(p) = 2\boldsymbol{g}^1_{\bar{s}}(p) - t(p)$ Use Prop

Now to the case $A^1 := k[x]$ and $A^2 = k[y_1, .., y_m]/J$ with $C_1 = k[x]/(x^{s+1})$ and C_2 a quotient of A^2 as above, so of socle type $\boldsymbol{t}$ and with $\{g_1, g_2, ..., g_\tau\}$ as a min. set of generators of C_2^*. Let $[\boldsymbol{g}]$ be the matrix $[g_1, g_2, ..., g_\tau]$.

Then there's a minimal set of generators of C_0^* of the form $\{f_1 + g_1, f_2 + g_2, ..., f_\tau + g_\tau\}$, with $1x\tau$-matrix $[\boldsymbol{f} + \boldsymbol{g}]$. As

$$1, x^{-1}, .., x^{-s+1} \in C_0^* \ , \quad (\mathit{recall}\ x(f_i + g_i) = xf_i)$$

we may take $f_1 = x^{-s}$ and $f_i = 0$ for $i > 1$. Also in the non-graded case we may take a minimal set of generators of C^* to be $\{x^{-s} + G_1, G_2, ..., G_\tau\}$ with $1x\tau$-matrix $[\boldsymbol{F} + \boldsymbol{G}]$. But $xg_i = 0$ and $xG_i = 0$. Thus both in the graded and filtered case there are $\tau - 1$ relations:

$$xg_i = 0 \quad \mathit{and} \quad xG_i = 0 \quad \mathit{for} \quad 2 \leq i \leq \tau \tag{11}$$

Thus there are matrices $M(\boldsymbol{f}+\boldsymbol{g})$ and $M(\boldsymbol{F}+\boldsymbol{G})$, both equal to:

$$\begin{bmatrix} 0 & 0 & .. & 0 \\ x & 0 & .. & 0 \\ 0 & x & .. & 0 \\ . & . & .. & . \\ 0 & 0 & .. & x \end{bmatrix}$$

such that $[\boldsymbol{f}+\boldsymbol{g}]M(\boldsymbol{f}+\boldsymbol{g})=0$ and $[\boldsymbol{F}+\boldsymbol{G}]M(\boldsymbol{F}+\boldsymbol{G})=0$. So there's a homogeneous complex

$$M := \bigoplus_{i=2}^{\tau} A(-e_i-1) \xrightarrow{M(\boldsymbol{f}+\boldsymbol{g})} \bigoplus_{i=1}^{\tau} A(-e_i) \xrightarrow{[\boldsymbol{f}+\boldsymbol{g}]} C_0^* \to 0 \qquad (12)$$

But $A_p = k[x]_p \oplus A_p^2$ for $p>0$ and mult. between elements of A_p^2 and entries of $M(\boldsymbol{f}+\boldsymbol{g})$ become 0. Thus (12) induces a complex:

$$0 \to \bigoplus_{i=2}^{\tau} k[x](-e_i - 1) \xrightarrow{M(\boldsymbol{f}+\boldsymbol{g})} \bigoplus_{i=1}^{\tau} A(-e_i) \xrightarrow{[\boldsymbol{f}+\boldsymbol{g}]} C_0^* \to 0 \qquad (13)$$

Here the map given by $M(\boldsymbol{f} + \boldsymbol{g})$ is injective (easy) as the last $\tau - 1$ rows in $M(\boldsymbol{f} + \boldsymbol{g})$ are zero except at one coordinate.

Proposition (Prop3: Useful presentation matrix of C_0)

(13) *is exact in degree* $\leq -b_1(A^2)$. *Thus, for* $b_1(A^2) \leq p < s$,

$$\dim(C_0)_p = g_{\bar{s}}(p) - e(p) \quad \textit{with} \quad e(p) := (\textstyle\sum_{q=p+1}^{s} \boldsymbol{t}(q)) - 1.$$

Also $(*)$: $\dim(C_0)_p = \boldsymbol{a}(p)$ *for* $p < v_0(A^2)$ (*and for* $p < b_1(A^2)$).

Proof To show (13) is exact in degree $d \leq -b_1(A^2)$ it suffices to see

$$\dim(C_0^*)_d + \sum_{i=2}^{\tau} \dim k[x](-e_i - 1)_d = \sum_{i=1}^{\tau} \boldsymbol{a}(-e_i)_d \qquad (14)$$

Put $p = -d$. As $\dim k[x]_q = 1$ for $q \geq 0$, the 1. sum equals $(\sum_{q=p+1}^{s} \boldsymbol{t}(q)) - 1 = e(p)$; the last sum $= \boldsymbol{g}_{\bar{s}}(p) = \sum_{q=p}^{s} \boldsymbol{t}(q)\boldsymbol{a}(q-p)$.

Now recall Prop.2 which implies

$\dim(C_0^*)_{-p} = \boldsymbol{h}_{\bar{s}}^{\mathrm{I}}(p) + 1$ for $0 < p < s$, where $\boldsymbol{h}_{\bar{s}}^{\mathrm{I}}(p) = \boldsymbol{h}_{\bar{s},A^2}^{\mathrm{I}}(p)$,

As the A^2-quotient C_2 is compressed, Remark 3 yields;

$$\dim(C_2)_p = \boldsymbol{a}^2(p) \quad \text{for} \quad p < v_0(A^2) \quad \text{and}$$

$$\dim(C_2)_p = \boldsymbol{g}_{\bar{s}}^2(p) := \textstyle\sum_{q=p}^{s} \boldsymbol{t}(q)\boldsymbol{a}^2(q-p) \text{ for } p \geq b_1(A^2).$$

Also $\dim(C_2)_p = \boldsymbol{h}_{\bar{s}}^{\mathrm{I}}(p)$ as C_2 is compressed. Combining, we get:

$\dim(C_0^*)_{-p} = \boldsymbol{a}^2(p) + 1 = \boldsymbol{a}(p)$ for $0 < p < v_0(A^2)$, so (*) holds, and

$$\dim(C_0^*)_{-p} = g_{\bar{s}}^2(p) + 1 \text{ for } s > p \geq b_1(A^2).$$

But $\boldsymbol{a}(q) - \boldsymbol{a}^2(q) = 1$ for $q > 0$ (and 0 for $q = 0$) which implies

$$\boldsymbol{g}_{\bar{s}}(p) - (\boldsymbol{g}_{\bar{s}}^2(p) + 1) = \textstyle\sum_{q=p+1}^{s} \boldsymbol{t}(q) - 1 = e(p). \text{ Thus}$$

$\dim(C_0^*)_{-p} = g_{\bar{s}}(p) - e(p)$ for $p \in [b_1(A^2), s\rangle$, so (13) is exact for $-p \leq -b_1(A^2)$ as (13) exact for $-p = -s$ is trivial. □

Recall (Remark 3) that if C is graded and compressed, then

$$\bigoplus_{q\in\mathbb{Z}} A(q)_p^{\oplus \boldsymbol{t}(q)} \xrightarrow{\cong} C_p^* \text{ for } p \leq -b_1 \quad \text{and} \quad A_q \xrightarrow{\cong} C_q \text{ for } q < b_1$$

And with C **filtered**, so possibly non-graded, the above, slightly reformulated, holds; e.g. let $N := \bigoplus_{q\in\mathbb{Z}} A(q)^{\oplus \boldsymbol{t}(q)}$. Then

$$N/F^{1-b_1}N \xrightarrow{\cong} C^*/F^{1-b_1}C^* \quad \text{and} \quad F^{1-b_1}C^* = F^{1-b_1}A^{\dagger}$$

if C is compressed. But a compressed filtered C has several other nice properties, e.g. in Tony's set-up, see [Iar84, Cor.3.8]:

C is compressed iff $G_\bullet(C)$ is compressed. Moreover if C is compressed, then $G_\bullet(C) = C_0$ where C_0 is generated by the initial forms of a minimal set of generators of C. Also the socle types of C and C_0 coincide.

There's a general lemma in [KK25], see (11.2), which was designed for proving $G_\bullet C^* = C_0^*$ and for comparing the socle types of C and C_0. Let's apply it directly in our situation. First composing the left module in (13) by $A \twoheadrightarrow k[x]$ as in the complex below, the exactness proved in Prop.3 yields (12) exact. As $[\boldsymbol{F} + \boldsymbol{G}]M(\boldsymbol{F} + \boldsymbol{G}) = 0$,

$$M := \bigoplus_{i=2}^{\tau} A(-e_i - 1) \xrightarrow{M(\boldsymbol{F}+\boldsymbol{G})} \bigoplus_{i=1}^{\tau} A(-e_i) \xrightarrow{[\boldsymbol{F}+\boldsymbol{G}]} C^* \hookrightarrow A^\dagger =: P$$

is a complex which extends (12). Now:

Lemma (Lem2: $G_\bullet C$ and socle type)

Let $M \xrightarrow{\mu} N \xrightarrow{\nu} P$ be a sequence of graded A-mod. Let $D := Im\,\nu$ and assume there's n with $M_p \xrightarrow{G_\bullet(\mu)} N_p \xrightarrow{G_\bullet(\nu)} P_p$ exact for all $p < n$ and with $\nu\mu M \subset F^{n+1}P$. Then $G_p D = Im\, G_p \nu$ for all $p \leq n$. Moreover if $(M/F^{p+1}M) \otimes_A k \to (N/F^{p+1}N) \otimes_A k$ vanishes for some $p < n$ then

$$((G_\bullet D) \otimes_A k)_p \xrightarrow{\cong} G_p(D \otimes_A k) \ .$$

Proposition (Prop4: $G_\bullet C$ and socle type for product rings)

With C^* and C_0^* as above, then

(i) $G_\bullet(C^*) = C_0^*$, *and*

(ii) $\boldsymbol{t}$ *is the socle type of both* C_0 *and* C.

Proof There's above a complex: $M \xrightarrow{M(\boldsymbol{F}+\boldsymbol{G})} N \xrightarrow{[\boldsymbol{F}+\boldsymbol{G}]} P$ that extends the the graded complex $M_p \xrightarrow{M(\boldsymbol{f}+\boldsymbol{g})} N_p \xrightarrow{[\boldsymbol{f}+\boldsymbol{g}]} P_p$ which by Proposition 3 is exact for $p \leq -b_1(A^2)$. Thus by Lemma 2

$$G_p C^* = (C_0^*)_p \text{ for all } p \leq 1 - b_1(A^2).$$

Moreover $\dim(C_0^*)_d = \boldsymbol{a}(-d)$ for $d > -b_1(A^2)$ by Prop.3. Hence

$$(C_0^*)_d = G_d(C^*) \quad \text{as} \quad A_d^\dagger = (C_0^*)_d \subset G_d(C^*) \subset A_d^\dagger.$$

It follows that $G_\bullet(C^*) = C_0^*$, as desired

By Lemma 2;

$$((G_\bullet C^*) \otimes_A k)_p \xrightarrow{\cong} G_p(C^* \otimes_A k) \quad \text{for } p \leq -b_1(A^2)$$

as the entries of $M(\boldsymbol{F} + \boldsymbol{G})$ belong to $F^1 A$. Thus for $p \leq -b_1(A^2)$;

(*) $$(C_0^* \otimes_A k)_p \xrightarrow{\cong} G_p(C^* \otimes_A k)\ .$$

But if $p > -b_1(A^2)$ then $(N \otimes k)_p = 0$ as $-b_1(A^2) \geq -\bar{s}$ and $N := \oplus_{q \in \mathbb{Z}} A(q)^{\oplus \boldsymbol{t}(q)}$. Moreover $N \twoheadrightarrow C_0^* = G_\bullet(C^*)$ is surjective, and so is

$$(G_\bullet C^*) \otimes_A k \twoheadrightarrow G_\bullet(C^* \otimes_A k)$$

by [KK25,(4.6)]. Thus both groups in **(*)** vanish. So **(*)** holds for all p which shows that the socle types of C of C_0 coincide.

□

Proposition (Prop5: tangent spaces)

Let C_0^* *be as above and let* $I = \ker(A \to C_0)$. *Then the tangent spaces,* $F^0 \operatorname{Hom}_A(I, C_0)$ *and* $F^0 \operatorname{Hom}_A(C_0^*, A^\dagger/C_0^*)$, *of* $\mathbb{F}\Psi_A^{\boldsymbol{h}}$ *and* $\mathbb{F}\Delta_{A^\dagger}^{\boldsymbol{h}^*}$ *at* C_0 *and* C_0^* *coincide and have dimension*

$$\sum_p \boldsymbol{t}(p)\boldsymbol{r}(p) \quad \text{where} \quad \boldsymbol{r}(p) = \sum_{q \le p} (\boldsymbol{a}(q) - \boldsymbol{h}_{C_0}(q)) .$$

Also the tangent spaces, $\operatorname{Hom}_A(I, C_0)_0$ *and* $\operatorname{Hom}_A(C_0^*, A^\dagger/C_0^*)_0$, *of* $\mathbb{H}\Psi_A^{\boldsymbol{h}}$ *and* $\mathbb{H}\Delta_{A^\dagger}^{\boldsymbol{h}^*}$ *coincide. They have dimension*

$$\sum_p \boldsymbol{t}(p)(\boldsymbol{a}(p) - \boldsymbol{h}_{C_0}(p)) .$$

Remark 4 Let C^*, C_0^* and I be as above and let $I_f = \ker(A \to C)$. Then the tangent spaces of $\mathbb{F}\Psi_A^{\boldsymbol{h}}$ and $\mathbb{F}\Delta_{A^\dagger}^{\boldsymbol{h}^*}$ given (with C for C_0) in the Prop.5 always coincide as their schemes are isomorphic by Thm.2. Similarly in the graded case. Note that these tangent (and obstruction) spaces are much studied by Jelisiejew, e.g. in Thm 4.2 in [Jel19]: J. Lond. Math. Soc.(2) 100 (2019). See also

[PGor98,Thm 1.10] and [KK25, (10.15)]

Proof To compute $\mathrm{Hom}_A(C_0^*, A^\dagger/C_0^*)_p$ for $p \geq 0$, let $Q := A^\dagger/C_0^*$. Recall $C_0^* = A(x^{e_1} + g_1, g_2, ..., g_\tau)$, so $x.(x^{-s} + g_1) \in C_0^*$. Thus the map

(*) $$Q_r \xrightarrow{.x} Q_{r+1} \text{ vanishes for } r \geq -s.$$

Note

$$\oplus_{i=2}^{\tau} A(-e_i - 1)_q \xrightarrow{M(\boldsymbol{f}+\boldsymbol{g})} \oplus_{i=1}^{\tau} A(-e_i)_q \xrightarrow{[\boldsymbol{f}+\boldsymbol{g}]} (C_0^*)_q \to 0$$

is exact in degree $q \leq -b_1(A^2)$; so replacing its leftmost term and arrow by $\oplus_{i=2}^{\tau+n} A(-p_i)$ and μ_0 where $p_i = e_i + 1$ for $2 \leq i \leq \tau$ and $p_i > -b_1(A^2)$ for $i > \tau$ (note there's no relation in deg $\leq -b_1(A^2)$ other than those generated by (11) as C_2^* has no relation in degree $\leq -b_1(A^2)$) we get, with $\nu := \mathrm{Hom}(\mu_0, Q)$, the diagram

$$\begin{array}{ccccc} \mathrm{Hom}_A(C_0^*, Q)_p & \hookrightarrow & \mathrm{Hom}_A(\oplus_{i=1}^{\tau} A(-e_i), Q)_p & \xrightarrow{\nu} & \mathrm{Hom}_A(\oplus_{i=2}^{\tau+n} A(-p_i), Q)_p \\ & & \downarrow\cong & \circ & \downarrow\cong \\ & & \oplus_{i=1}^{\tau} Q(e_i)_p & \xrightarrow{\nu} & \oplus_{i=2}^{\tau+n} Q(p_i)_p \end{array}$$

We claim $\nu = 0$

Claim: $\nu = 0$. Indeed to show $\nu\phi = 0$ for any $\phi \in \oplus_{i=1}^{\tau} Q(e_i)_p$, replace ν and ϕ by matrices $[\nu]$ and $[q_1, .., q_\tau]^{tr}$. Note $[\nu]$ is a $(\tau + n)x\tau$ matrix whose first $(\tau - 1)$ rows are of the form $[0, ..0, x, 0, ..0]$ as they are given by the columns of $M(\boldsymbol{f} + \boldsymbol{g})$. Thus in the product $[\nu][q_1, .., q_\tau]^{tr}$ they become 0 by **(*)** as $e_i + p \geq -s$. For the other rows, the entries of the above product are of degree

$$p_i + p > -b_1(A^2) + p \geq -b_1(A^2) \quad \textit{for } p \geq 0 \ .$$

But $Q_r = 0$ for $r > -b_1(A^2)$ as $\dim(C_0^*)_d = \boldsymbol{a}(-d)$ by Prop.3. Thus $\nu = 0$, which implies

$$\mathrm{Hom}_A(C_0^*, Q)_p \cong \oplus_{i=1}^{\tau} Q(e_i)_p.$$

Then Prop.5 follows by counting dimensions of the free module $\oplus_{i=1}^{\tau} Q(e_i)_p \cong \oplus_{q\in\mathbb{Z}} Q(-q)_p^{\oplus \boldsymbol{t}(q)}$, using $\dim Q_{-v} = \boldsymbol{a}(v) - \boldsymbol{h}_{C_0}(v)$. So $\dim \oplus_{i=1}^{\tau} Q(e_i)_p = \sum_q \boldsymbol{t}(q)(\boldsymbol{a}(q-p) - \boldsymbol{h}_{C_0}(q-p))$. Then take $p = 0$ (resp. $\sum_{p\geq 0}$) in the graded (resp. filtered) case. □

The results above, in particular Proposition 2, fit with the theory developed in [KK25, (7.6)-(7.10)]. Indeed applying Prop. 2(4) to C_0 as well as to all $(\Delta^m C_0^*)^*$ (with Hilbert function $\boldsymbol{h}_m$), assuming there C_i I-compressed for $i = 1, 2$, we get that $\{\boldsymbol{h}_m\}$ is **recursively maximal** and $\boldsymbol{t}$ quasipermissible in the sense of [KK25,(7.6)-(7.7)]. Thus $C_0 \in \boldsymbol{H}\Lambda_A^{\{\boldsymbol{h}_m\}}$ is a closed point of $\mathbb{H}\Lambda_A^{\{\boldsymbol{h}_m\}}$ below where $S := Spec(k)$ with k a noetherian ring.

The theorem is inspired by [Iar84, Prop. 3.6].

Theorem (Thm3, the scheme of recursively maximal quotients)

If $\boldsymbol{t}$ is quasi-permissible and S is reduced and irreducible, then there exists a recursively maximal set $\{\boldsymbol{h}_m\}$ for $\boldsymbol{t}$ and T/S; moreover, for any such set $\{\boldsymbol{h}_m\}$, then $\mathbb{H}\Lambda_A^{\{\boldsymbol{h}_m\}}$ is nonempty, reduced, and irreducible, and it's covered by open subschemes, with each one isomorphic to an open subscheme of the affine space over S of fiber dimension $\boldsymbol{H}$ where $\boldsymbol{H} := \sum_p \boldsymbol{t}(p)(\boldsymbol{a}(p) - \boldsymbol{h}_{\bar{s}}(p))$.

But Prop. 5 (and Prop. 4) indicates that $\mathbb{F}\Lambda_A^{\{\boldsymbol{h}_m\}}$ may be similarly nice. i.e. that the folowing can be true

Conjecture Set

$$\boldsymbol{F} := \sum_p \boldsymbol{t}(p)\boldsymbol{r}(p) \quad \text{where} \quad \boldsymbol{r}(p) := \sum_{q \le p}(\boldsymbol{a}(q) - \boldsymbol{h}_{\bar{s}}(q)),$$

and assume that $\boldsymbol{t}$ is permissible. Then for any recursively maximal set $\{\boldsymbol{h}_m\}$ for $\boldsymbol{t}$ and T/S, $\mathbb{F}\Lambda_A^{\{\boldsymbol{h}_m\}}$ is covered by open subschemes, each one isomorphic to an open subscheme of the affine space over S of fiber dim. $\boldsymbol{F}$. Also $\mathbb{F}\Lambda_A^{\{\boldsymbol{h}_m\}}$ is irreducible if S is irreducible.

See [KK25,(10.12)] for $\boldsymbol{h}_m = \boldsymbol{h}_m^{\mathrm{I}}$ where we also needed to put some mild assumptions on A too.

Thanks for listening