Cox-Gorenstein Algebras

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Outline of the talk

1. Toric varieties and Cox rings
2. Artinian G-graded algebras and Cox-Gorenstein algebras
3. The codimension one Conjecture and our result
4. Toric Lefschetz properties and the Hessian criterion

Toric varieties and the Cox ring

Definition (Toric varieties)

A toric variety is an irreducible variety X containing an algebraic torus $T \simeq (\mathbb{C}^*)^d$ as a Zariski open subset such that the action, $T \times T \to T$, of T on itself extends to an algebraic action of T on X.

There is a way to construct a toric variety from a fan.

Definition (The Cox ring of a toric variety)

Let X be a complete toric variety. The Cox ring of X is the graded ring

$$\mathrm{Cox}(X) := \bigoplus_{[D] \in \mathrm{Cl}(X)} H^0(\mathcal{O}_X(D))$$

The grading in the Cox ring

Let $X = \mathbb{P}_\Sigma$ be a complete toric variety X. Its Cox ring, $S = \mathrm{Cox}(X)$ is a polynomial ring that has one variable for each ray of the fan. S has a natural grading given by the class group $G = \mathrm{Cl}(X)$, which is a finitely generated abelian group. There is a natural partial order in G that makes the effective divisors positive. Moreover, every finitely generated abelian group can be presented as the class group of some toric variety.

The converse

Conversely, given a finitely generated partially ordered abelian group G and a G-grading over a polynomial ring S, we can recover the toric variety when the rank of G is less than or equal to the number of variables of S.

The algorithm

1. Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring graded by a finite-generated Abelian group $G = \mathbb{Z}^r \oplus \mathbb{Z}_{m_1} \oplus \ldots \oplus \mathbb{Z}_{m_N}$. The grading defines matrices P_i with $i = 0, \ldots, N$.
2. Let $\Gamma = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)$ and embed it into $\mathbb{T}^n$ as prescribed by the matrices P_i.
3. Following this prescription, one first gets a polytope, then a fan Σ, and from it the irrelevant ideal I.
4. The toric variety X is given by the fan Σ. It is the GIT quotient:

$$X = \mathbb{A}^n /\!/ \Gamma = \frac{\mathbb{A}^n - V(I)}{\Gamma}.$$

Artinian G-graded algebras

Let $(G, +, \leq)$ be a partially ordered Abelian group, let $\mathbb{K}$ be a field of characteristic zero, and let R be a $\mathbb{K}$-algebra.

Definition (G-graded algebras)

A G-grading in R is a decomposition of R as a direct sum of $\mathbb{K}$-vector spaces $R = \oplus_{g \in G} R_g$ such that the product in R satisfies $R_g R_h \subset R_{g+h}$. We always consider $R_0 = \mathbb{K}$ and we say that R is finitely graded if $R_g = 0$ for all but finitely many $g \in G$. Given a homogeneous element $f \in R_g \setminus 0$, we denote $\deg(f) = g \in G$.We say that the algebra is an Artinian G-graded algebra if it is an Artin ring, neglecting the grading.

Hasse-Hilbert diagram

Let A be an Artinian G-graded $\mathbb{K}$-algebra. The Hilbert function of A

$$\mathrm{HF}_A : G \to \mathbb{Z}_+$$

is defined as $\mathrm{HF}_A(g) = h_g := \dim A_g$. To properly encode the structural information contained in the Hilbert function we introduce the Hasse-Hilbert diagram of A defined as a vertex-weighted directed graph structure over the covering graph of G where the weight of a vertex g is its Hilbert function h_g.

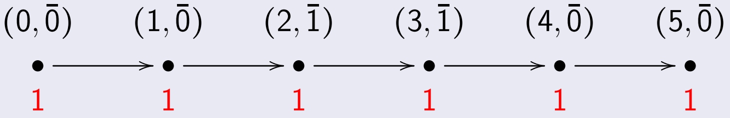
Extremal elements in the diagram

By definition of the covering graph of a partial order, its vertex set is the POSET G, and two vertices $g, h \in G$ are connected if they are immediate neighbors, that is, they are comparable and there is no other comparable element between them. As usual, a maximal element in a POSET X is an element $x \in X$ for which it does not exist $y \in X$ such that $x \leq y$ with $x \neq y$. We say that a maximal element $x \in X$ is the greatest element in X if $y \leq x$ for all $y \in X$.

Example (1)

Let $S = \mathbb{C}[x, y, z]$ be the polynomial ring with a $G = \mathbb{Z} \oplus \mathbb{Z}_2$-grading given by $\deg(x) = (1, \bar{1})$, $\deg(y) = (1, \bar{0})$ and $\deg(z) = (2, \bar{1})$. Let $I = (x, y^2, z^3) \subset S$ and $A = S/I$. It is easy to see that A is an Artinian G-graded algebra. The Hasse-Hilbert diagram of A is linear and we can write

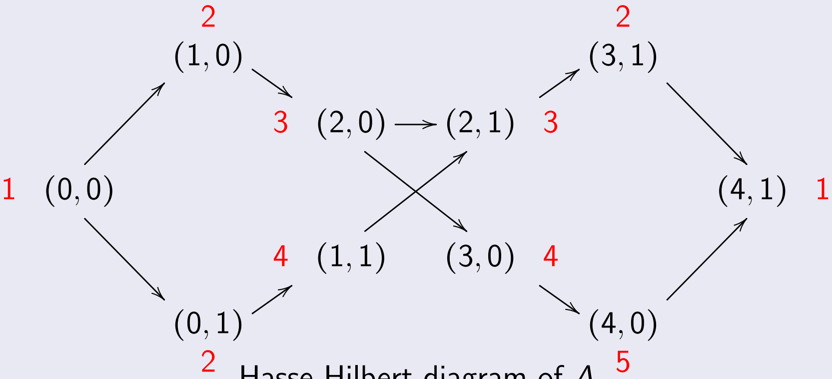
$$A = A_{(0,\bar{0})} \oplus A_{(1,\bar{0})} \oplus A_{(2,\bar{1})} \oplus A_{(3,\bar{1})} \oplus A_{(4,\bar{0})} \oplus A_{(5,\bar{0})}.$$



Hasse-Hilbert diagram of A

Example (2)

Let $S = \mathbb{C}[x, y, u, v]$ be $\mathbb{Z}^2$ graded $\deg(x) = \deg(y) = (1, 0)$, $\deg(u) = \deg(v) = (0, 1)$. Let $A = S/I$ with $I = (S_{(0,2)}, S_{(4,0)}, x^2u - y^2v, x^2v, y^2u)$.



Hasse-Hilbert diagram of A

Cox-Gorenstein algebras

Definition (Cox-Gorenstein algebras)

Let $A = \oplus_{g\in G} A_g$ be an Artinian G-graded $\mathbb{K}$-algebra of finite type. We say that A is Cox-Gorenstein if there is $\rho \in G$ such that $\mathrm{Soc}(A) := (0 : \mathfrak{m}) = A_\rho \simeq \mathbb{K}$.

Remark

Notice that the socle of an arbitrary G-graded Artinian algebra is always a maximal element in the POSET. On the other hand, when the algebra has the greatest element, then its degree is called the socle degree of the algebra. This is always the case when A is Cox-Gorenstein.

Poincaré duality algebras

Definition (Poincaré duality algebras)

Let $A = \oplus_{g \in G} A_g$ be an Artinian G-graded $\mathbb{K}$-algebra of finite type. We say that A has the Poincaré duality if there is a maximal element $\rho \in G$ such that $A_\rho \simeq \mathbb{K}$ and the multiplication maps

$$A_g \times A_{\rho - g} \to A_\rho$$

are perfect pairings, whenever A_g and $A_{\rho-g}$ are non-zero. Moreover, if one is zero, then the other one is also zero.

Example

The algebra in Example 1 is Cox-Gorenstein while the one in Example 2 is not.

Characterization of Cox-Gorenstein algebras

Theorem

Let A be a G-graded Artinian $\mathbb{K}$-algebra of finite type. A is Cox-Gorenstein if and only if A has the Poincaré duality.

Theorem (Macaulay)

Let $I \subset Q = \mathbb{K}[X_1, \ldots, X_n]$ be a G-homogeneous ideal such that $A = Q/I = \oplus_{g \in G} A_g$ is an Artinian G-graded $\mathbb{K}$-algebra. Then A is Cox-Gorenstein of socle degree ρ if and only if there is $f \in S_\rho$ such that $I = \mathrm{Ann}(f)$.

Macaulay's CI Theorem

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. As motivation, let $X = V(f) \subset \mathbb{P}^n$ be a smooth projective hypersurface and let $J \subset S = \mathbb{K}[x_0, \ldots, x_n]$ be its Jacobian ideal; then the so called Milnor algebra $A = S/J$ is an Artinian complete intersection.

Theorem (Macaulay)

Let $f_1, \ldots, f_n \in S$ be a sequence of homogeneous polynomials of degree $\deg(f_i) = d_i$, not vanishing simultaneously. Then $f_1, \ldots, f_n$ is a regular sequence, that is $A = \mathbb{K}[x_0, \ldots, x_n]/(f_1, \ldots, f_n)$ is an Artinian complete intersection. In particular A is Gorenstein of socle degree $\sum d_i - n - 1$.

The toric Jacobian ideal

Let $X = \mathbb{P}_\Sigma$ be a complete toric variety and let $S = Cox(X) = \mathbb{K}[x_1, \dots, x_n]$ be its Cox ring. Let $f \in S_\beta$.

Definition

The toric Jacobian ideal of f is

$$J_0(f) = (x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n}).$$

Theorem (Bruzzo, Montoya-[BM])

Let α be an ample Cartier class. The toric Milnor algebra $A = S/J_0$ associated to $f \in S_\alpha$ is Cox-Gorenstein if and only if the Picard number $r = 1$. In this case, the socle degree is $\rho = n\alpha - \beta$ where β is the anticanonical class of X.

An example

Example (3)

Let $G = \mathbb{Z} \oplus \mathbb{Z}_2$ and consider $Q = \mathbb{C}[X, Y, Z]$ G-graded by $\deg(X) = (1, \overline{1})$, $\deg(Y) = (1, \overline{0})$ and $\deg(Z) = (2, \overline{1})$. Let J_0 be the toric Jacobian ideal of $F = X^4 + Y^4 + Z^2 \in Q_{(4,\overline{0})}$, that is, $J_0 = (X^4, Y^4, Z^2)$. Since $(X^4, Y^4, Z^2) \subset Q$ is a complete intersection, $A = Q/I$ is a Cox-Gorenstein algebra. In fact, considering Q as a ring of differential operators, we get $I = \mathrm{Ann}(f)$ with $f = x^3y^3z \in \mathbb{C}[x, y, z]$. The socle degree of A is $\mathrm{soc.deg}(A) = \deg(f) = (8, \overline{0})$.

Remark

The only toric varieties $X = \mathbb{P}_\Sigma$ having Picard number $r = 1$ are weighted projective spaces and fake weighted projective spaces.

The critical degree $\rho \in G$

Let X be a complete simplicial toric variety and let $f_0, \ldots, f_n \in B(\Sigma)$ be homogeneous polynomials wich do not vanish simultaneously on X of degree $\deg(f_i) = \alpha_1$. Let β be the sum of the degree of the variables, it is well known that β is the anticanonical class in X.

Definition

The critical degree of this configuration is $\rho = \sum \alpha_i - \beta$.

The codimension one Conjecture of Cattani-Cox-Dickeinstein

Conjecture (Cattani, Cox, Dickeinstein-[CCD])

If $A = S/(f_0, \dots, f_n)$ *then:*

$$\dim_{\mathbb{C}} A_\rho = 1.$$

Remark

In the same paper the authors prove that The codimension one conjecture implies that A_ρ is a maximal element, that is A_ρ is inside the socle of A. Some particular cases of the conjecture have been proved so far (for instance, Villaflor in [V] and Cox-Dickeinstein in [CD] proved special cases), but the general conjecture is still open.

Our special case

Theorem (Bruzzo,-, Holanda, Montoya)

Let $X = \mathbb{P}_\Sigma$ be a d-dimensional complete simplicial toric variety with Cox ring $S = \mathbb{C}[x_1, \dots, x_n]$, and let $f_i \in S_{\alpha_i}$, for $i = 0, \dots, d$, be homogeneous polynomials, and all α_i nef and the polytope P_η is full dimensional for all $\eta \in \bigcup_j \mathbf{X}_j$. Let us assume that the f_i's do not vanish simultaneously on $\mathbb{P}_\Sigma$. Let $\rho = \sum_{i=0}^{d} \alpha_i - \beta$ be the critical degree. If $A = S/(f_0, \dots, f_d)$ then

1. $A_\rho \simeq \mathbb{C}$;
2. $x_i A_\rho = 0$ for $i = 1, \dots, n$.

Moreover, if the Picard number $r = 1$, then $(f_0, \dots, f_d)$ is a complete intersection, A is a Cox-Gorenstein $\mathbb{C}$-algebra of socle degree ρ.

Linearly consecutive and linear comparable pieces

Let $A = Q/I = \oplus_{g\in G} A_g$ be a G-graded Artinian $\mathbb{K}$-algebra and let $\mathcal{L} = \langle X_1, \dots, X_n\rangle \subseteq A$ be the $\mathbb{K}$-linear subspace generated by the class of the variables in Q. We denote $\mathcal{L}_g = \mathcal{L} \cap A_g$. We say that two graded pieces of A let us say A_g and A_h are

- linearly consecutive if $g \le h$ and $\mathcal{L}_{h-g} \ne 0$;
- linearly comparable if $g \le h$ and there is $L \in \mathcal{L}_l$ such that $h = g + kl$ for some $k \in \mathbb{Z}_+$.

Linearly consecutive and linear comparable pieces

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- linearly comparable if $g \leq h$ and there is $L \in \mathcal{L}_l$ such that $h = g + kl$ for some $k \in \mathbb{Z}_+$.

Toric Lefschetz properties

Definition

We say that A has the Toric Weak Lefschetz property (TWLP) if for every linearly consecutive pieces A_g and A_h there is a linear element $L \in \mathcal{L}_{h-g}$ such that the $\mathbb{K}$-linear multiplication map $\bullet L : A_g \to A_h$ has maximal rank.

Definition

We say that A has the Toric Strong Lefschetz property (TSLP) if for every linearly comparable pieces A_g and A_h there is a linear element $L \in \mathcal{L}_l$ such that $h = g + kl$ and the $\mathbb{K}$-linear multiplication map $\bullet L^k : A_g \to A_h$ has maximal rank.

Example (4)

Consider $S = \mathbb{K}[x, y, u, v]$ and $G = \mathbb{Z} \oplus \mathbb{Z}$ and a G-grading given by $\deg(x) = \deg(y) = (1, 0)$ and $\deg(u) = \deg(v) = (0, 1)$. Let $f \in S_{(2,3)}$ be given by $f = x^2u^3 + y^2v^3$. Let $Q = \mathbb{K}[X, Y, U, V]$ be the ring of differential operators acting on S and let $I = \mathrm{Ann}(f) \in Q$ such that $A = Q/I = \oplus_{k=0}^{5} A_k$ is a Cox-Gorenstein algebra and $A_k = \oplus_{i+j=k} A_{ij}$. Here we are considering the total degree as a $\mathbb{Z}$-grading for A, that is, $\phi(m, n) = m + n$. We have two spaces of homogeneous linear elements, $\mathcal{L}_{(1,0)} =< X, Y >$ and $\mathcal{L}_{(0,1)} =< U, V >$.

Example

(cont.)
From the standard $\mathbb{Z}$-grading, we know that the WLP can be verified in the middle, that is, from A_2 to A_3. Since $A_2 = A_{(2,0)} \oplus A_{(1,1)} \oplus A_{(0,2)}$ and $A_3 = A_{(2,1)} \oplus A_{(1,2)} \oplus A_{(0,3)}$ we see that $L = U + V$ can be used as ϕ-linear element simultaneously for $\bullet L : A_{(2,0)} \to A_{(2,1)}$, $\bullet L : A_{(1,1)} \to A_{(1,2)}$ and $\bullet L : A_{(0,2)} \to A_{(0,3)}$. We know that $A_{(2,0)} =< X^2, Y^2 >$, $A_{2,1} =< X^2U, Y^2V >$ and the multiplication map by L is actually an isomorphism. In the same way we can verify that $\bullet L : A_{(1,1)} \to A_{(1,2)}$ and $\bullet L : A_{(0,2)} \to A_{(0,3)}$ are isomorphism and A has the WLP with the total grading. It also have the TWLP.

Example (5)

Let $S = \mathbb{K}[x, y, z]$ be $\mathbb{Z}$-graded by $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$. Let $f \in S_4$ given by $f = x^4 + y^4 + z^2$. In the dual $Q = \mathbb{K}[X, Y, Z]$ we obtain $\mathrm{Ann}(f) = (XY, XZ, YZ, X^5, Y^5, Z^3)$. Let $A = Q/I$ be the Artinian Cox-Gorenstein algebra associated to f. We have $A = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus A_4$ with $A_1 =< X, Y >$, $A_1 =< X^2, Y^2, Z >$, $A_3 =< X^3, Y^3 >$, and $A_4 =< X^4 >$. It is easy to verify that A has the TSLP with $L = X + Y$.

Phi-linearity

Let $A = \oplus_{g \in G} A_g$ be a Artinian G-graded $\mathbb{K}$-algebra. Let $\phi \in \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q})$ and let $\mathcal{L} =< X_1, \dots, X_s > \subset A$ be the $\mathbb{K}$-linear subspace generated by the class of the variables in Q. We say that a homogeneous element $L \in \mathcal{L}_g := \mathcal{L} \cap A_g$ is ϕ-linear if $\phi(g) = 1$.

Toric Euler identities

Lema

Suppose that S is G-graded with $\deg(x_i) = g_i$ and let $f \in S_g$. For every $\phi \in \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q})$, there is a generalized Euler relation

$$\sum_{i=1}^{n} \phi(g_i) x_i \frac{\partial f}{\partial x_i} = \phi(g) \cdot f$$

Proposition (Bruzzo,-, Holanda, Montoya)

Let $L = a_1 X_1 + \ldots + a_m X_m \in \mathcal{L}_g$ be a ϕ-linear element and f a homogeneous polynomial of degree $\rho \in G$ such that $\phi(\rho) \in \mathbb{Z}_+$ then,

$$L^{\phi(\rho)} f = \phi(\rho)! f(a) \quad \text{where} \quad a = (a_1, \ldots, a_m, 0, \ldots, 0)$$

Toric Hessian Criterion

Definition

Let $\mathcal{B} = \{\beta_1, \ldots, \beta_s\}$ and $\mathcal{C} = \{\gamma_1, \ldots, \gamma_t\}$ be $\mathbb{K}$-basis of A_{g_i} and A_{g_j} respectively. The toric mixed Hessian of f with mixed order (β, γ) is

$$\mathrm{Hess}_f^{(\mathcal{B},\mathcal{C})} := [\beta_i \circ \gamma_j(f)]$$

Remark

As usual, the definition of the Hessian depends on the basis, but its rank does not depend.

Toric Hessian Criterion

Theorem (Bruzzo,-, Holanda, Montoya)

Let $A = Q/I$ with $I = \mathrm{Ann}(f)$ be a Artinian G-graded Cox-Gorenstein $\mathbb{K}$ algebra of socle degree $\rho \in G$. Let A_g and A_h be two linearly comparable graded pieces of A such that $g < h$ and $h = g + kl$ with $L \in \mathcal{L}_l$ a ϕ-linear element $L = a_1X_1 + \ldots + a_mX_m$. Then the matrix of the $\mathbb{K}$-linear map $\bullet L^k : A_g \to A_h$ with respect to the basis $\mathcal{B}$ and $\mathcal{C}$ is:

$$\left[\cdot L^k\right]_{\mathcal{B}}^{\mathcal{C}} = k! \cdot \mathrm{Hess}_f^{(\mathcal{C}^*, \mathcal{B})}(a)$$

where $a = (a_1, \ldots, a_m, 0, \ldots, 0)$. In particular, the mixed Hessians controls both, TWLP and TSLP.

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