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Regularity of tangential decompositions

Jordan Types of Artinian Algebras and Geometry of Punctual Hilbert Schemes - Celebrating the 80th birthday of Anthony Iarrobino

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Based on a joint work with A. Bernardi e A. Oneto

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Main theme

$\Bbbk$: an algebraically closed field of characteristic 0.

$\mathcal{S} = \oplus_{d=0}^{\infty} \mathcal{S}_d = \Bbbk[x_0, \ldots, x_n]$ graded polynomial ring.

(Normalized) Generalized Additive Decomposition (GAD)

An expression of $F \in \mathcal{S}_d$ supported at $\{L_i\}_{i \in \{1 \ldots s\}} \subset \mathcal{S}_1$ as

$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i, \quad \text{with} \quad L_i \nmid G_i \in \mathcal{S}_{k_i} \;\; \text{and} \;\; \mathbb{P}L_i \neq \mathbb{P}L_j \text{ for } i \neq j.$$

Goal

Minimize its *rank*, for some weight function $wt(\cdot)$:

$$\mathrm{rk}(F) = wt(L_1^{d-k_1} G_1) + \cdots + wt(L_s^{d-k_s} G_s).$$

A. Iarrobino, V. Kanev, *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Springer Berlin, 1999.

Related problems

Related problems

- Waring decomposition(*) $[L^d]$
- Tangential decomposition(*) $[L_1^{d-1}L_2]$
- Slice decomposition(‡) $[L\,G]$
- Chow-Veronese decomposition(†) $[L_1\cdots L_d]$
- GAD (cactus/scheme length*) decomposition(*) $[L^{d-k}G]$
- Strength decomposition(‡) $[G_1G_2]$

(†) E. Arrondo, A. Bernardi, *On the variety parameterizing completely decomposable polynomials*, J. Pure Appl. Algebra 215, 2011, pp. 201–220.

(*) A. Bernardi, D. Taufer, *Waring, tangential and cactus decompositions*, J. Math. Pures Appl. 143 (2020), pp. 1–30.

(‡) A. Bik, J. Draisma, R.H. Eggermont, *Polynomials and tensors of bounded strength*, Commun. Contemp. Math. 21(7), 2019.

Weighting via apolar schemes

$\mathcal{R} = \oplus_{d=0}^{\infty} \mathcal{R}_d = \Bbbk[y_0, \ldots, y_n]$ graded polynomial ring, acting on $\mathcal{S}$ via

$$\mathcal{R}_k \times \mathcal{S}_d \to \mathcal{S}_{d-k}, \quad (G, F) \mapsto G \circ F.$$

Apolar schemes

A scheme $Z \subseteq \mathbb{P}^n(\Bbbk)$ is *apolar* to $F \in \mathcal{S}_d$ if

$$I(Z) \subseteq \mathrm{Ann}(F) = \{G \in \mathcal{R} \mid G \circ F = 0\}. \qquad [F^\perp]$$

(Reduced) Apolarity Lemma

The following are equivalent:

- $F = L_1^d + \cdots + L_s^d$,
- there is an ideal $I \subseteq \mathrm{Ann}(F)$ defining s simple points in $\mathbb{P}^n(\Bbbk)$.

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GADs and apolar schemes [🗎, Thm. 5.3]

The following are equivalent:

- $F = L_1^{d-k_1} G_1 + \cdots + L_s^{d-k_s} G_s$,
- there is an ideal $I \subseteq \mathrm{Ann}(F)$ defining s points in $\mathbb{P}^n(\Bbbk)$, whose components are contained in $(d - k_i)$-fat points.

🗎 A. Iarrobino, V. Kanev, *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Springer Berlin, 1999.

Weighting by apolarity

GAD $\rightsquigarrow$ 0-dim. scheme

There is an explicit construction through *natural apolar schemes*:

$$F = L_1^{d-k_1}G_1 + \cdots + L_s^{d-k_s}G_s \quad \rightsquigarrow \quad Z_{L_1^{d-k_1}G_1, L_1} \cup \cdots \cup Z_{L_s^{d-k_s}G_s, L_s}.$$

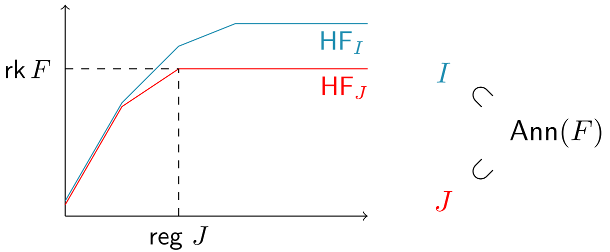
Weights

$$wt(L^{d-k}G) = \mathsf{len}(Z_{L^{d-k}G,L}) = \deg\left(I(Z_{L^{d-k}G,L})\right).$$

A. Bernardi, J. Jelisiejew, P. M. Marques, K. Ranestad, *On polynomials with given Hilbert function and applications*, Collect. Math. 69, 2018, pp. 39–64.

An implementation (M2 and Magma): https://github.com/DTaufer/SchemesEvincedByGADs

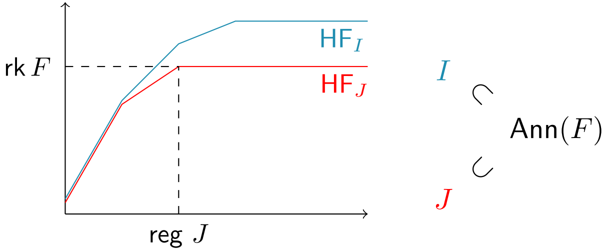
Hilbert functions



Minimal apolar schemes formulation

Finding a minimal (saturated) homogeneous 0(-affine)-dimensional ideal $I \subseteq$ Ann(F).

Hilbert functions



Minimal apolar schemes formulation

Finding a minimal (saturated) homogeneous 0(-affine)-dimensional locally-Gorenstein ideal $I \subseteq \mathrm{Ann}(F)$.

W. Buczyńska, J. Buczyński, *Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes*, J. Algebr. Geom. 23(1), 2014, pp. 63–90.

Today's focus

(Open) question

Let $F \in \mathcal{S}_d$ and $I \subseteq \mathrm{Ann}(F)$ be 0-dimensional and of minimal degree. Can we bound $\operatorname{reg} I$ in terms of d?

Tangential decompositions (schemes of 2-jets)

$$\mathcal{J}_2(F) = \{I = I_1 \cap \cdots \cap I_s \mid I \subseteq \mathrm{Ann}(F) \text{ and } \deg(I_i) \leqslant 2\}.$$

Results [📄, Prop. 5.9, Ex. 5.10]

Let $F \in \mathcal{S}_d$ and $I \in \mathcal{J}_2(F)$.

- If I is of minimal length in $\mathcal{J}_2(F)$, then $\operatorname{reg} I \leqslant d$.
- If I is *irredundant* to F, then $\operatorname{reg} I$ may exceed d.

📄 A. Bernardi, A. Oneto, D. Taufer, *On schemes evinced by generalized additive decompositions and their regularity*, J. Math. Pures Appl. 188, 2024, pp. 446–469.

Non-regularity $\rightsquigarrow$ relations

Let $L \in \mathcal{S}_1$ and $\mathcal{D}_L = L^\perp \cap \mathcal{R}_1$. We define the $\Bbbk$-vector space

$$\mathcal{D}_L^e(F) = \langle H \circ F \mid H \in \mathcal{D}_L^e \rangle \subseteq \mathcal{S}_{d-e}.$$

Degree-d part [🗎, Rmk. 3]

$$I(Z_{F,L})_d^\perp = \left\langle \sum_{i=0}^{d} L^e \cdot \mathcal{D}_L^e(F) \right\rangle \subseteq \mathcal{S}_d.$$

Non-regularity in degree d of $Z = Z_{F_1,L_1} \cup \cdots \cup Z_{F_s,L_s}$ means

$$\dim_\Bbbk I(Z)_d^\perp < \dim_\Bbbk I(Z_{F_1,L_1})_d^\perp + \cdots + \dim_\Bbbk I(Z_{F_s,L_s})_d^\perp,$$

hence the vector spaces $I(Z_{F_i,L_i})_d^\perp$ intersect.

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hence the vector spaces $I(Z_{F_i,L_i})_d^\perp$ intersect. $\leftarrow$ "relation"

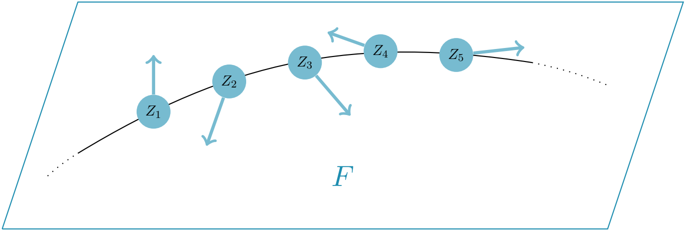
🗎 A. Bernardi, J. Jelisiejew, P. M. Marques, K. Ranestad, *On polynomials with given Hilbert function and applications*, Collect. Math. 69, 2018, pp. 39–64.

Irredundancy fails

Let Z be the scheme evinced by the GAD of $F \in \mathbb{C}[x, y, z_1, z_2, z_3]_3$:

$$x^2z_1+y^2z_2+(x+y)^2z_3+(x-y)^2(z_1-3z_2-2z_3)+(x+2y)^2(z_1+z_2+z_3).$$

Z is the union of five 2-jets $Z_1, \ldots, Z_5$.



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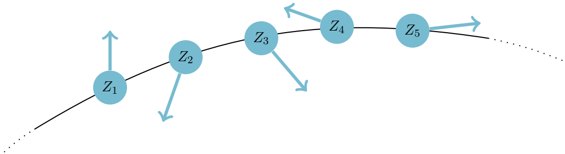


Its Hilbert function is $[1, 5, 8, 9, 10, 10, \ldots]$. Hence, we have a relation:

$$3x^3 - 6y^3 - 3(x+y)^3 - (x-y)^3 + (x+2y)^3 = 0.$$

One can directly check that Z is irredundant for F, i.e. **[T]**

$$Z' \subsetneq Z \implies I(Z') \subsetneq \mathrm{Ann}(F).$$

Improving the GAD

$$F = \sum_{i=1}^{s} L_i^{d-1} G_i + \sum_{i=s+1}^{r} L_i^s \in \mathcal{S}_d.$$

Since Z is not d-regular, we need to have a relation in

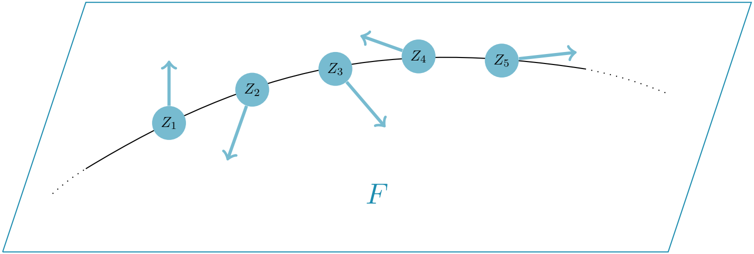
$$I(Z)_d^{\perp} = \langle L_i^d, L_j^{d-1} G_j \rangle_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}}.$$

"External" relations [🗎, Prop. 5.2]

Let Z be the scheme evinced by the GAD: $F = \sum_{i=1}^{s} L_i^{d-k_i} G_i \in \mathcal{S}_d$. If there is a relation in $I(Z)_d^{\perp}$ involving $L_i^{d-k_i} G_i$, for any $1 \leqslant i \leqslant s$, then Z is redundant to F.

🗎 A. Bernardi, A. Oneto, D. Taufer, *On schemes evinced by generalized additive decompositions and their regularity*, J. Math. Pures Appl. 188, 2024, pp. 446–469.

Improving the GAD



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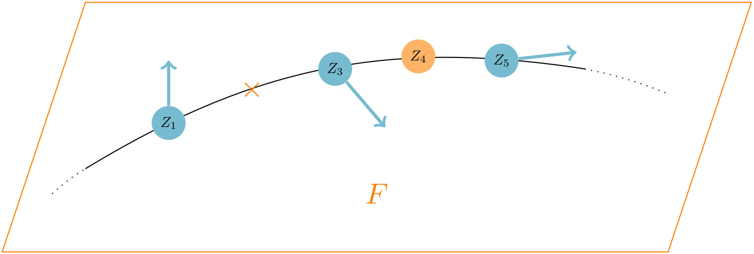

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Improving the GAD

$$F = \sum_{i=1}^{s} {L_i}^{d-1} G_i + \sum_{i=s+1}^{r} L_i^s \in \mathcal{S}_d.$$

If d-irregular but without "external" relations, then **[T]**

$$\sum_{i=1}^{s} \lambda_i {L_i}^{d} = 0,$$

$$\Downarrow \partial$$

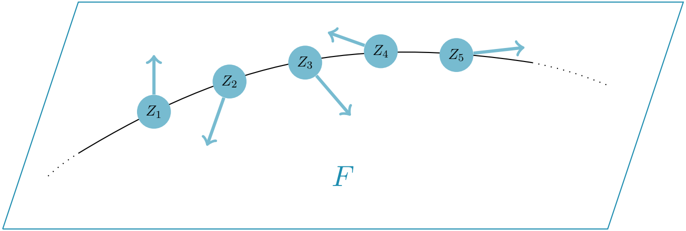
$$\sum_{i=1}^{s} \lambda_i' {L_i}^{d-1} = 0.$$

Improving the GAD

Can substitute (say) L_1:

$$F = \sum_{i=2}^{s} L_i^{d-1}(G_i + \lambda_i' G_1) + \sum_{i=s+1}^{r} L_i^s \in \mathcal{S}_d.$$

This new GAD evinces a shorter scheme, again in $\mathcal{J}_2(F)$. **[T]**



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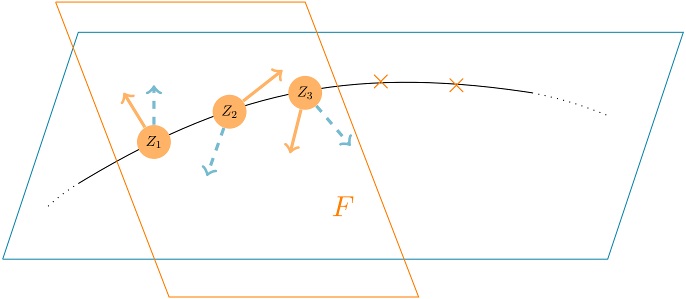

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Applied to our example

From the relation

$$(x-y)^2 = 2x^2 + 2y^2 - (x+y)^2$$

we produce a better GAD:

$$\begin{aligned} F = {} & x^2(3z_1 - 6z_2 - 4z_3) + y^2(2z_1 - 5z_2 - 4z_3) \\ & + (x+y)^2(-z_1 + 3z_2 + 3z_3) + (x+2y)^2(z_1 + z_2 + z_3). \end{aligned}$$

From the other relation

$$(x+2y)^2 = 2(x+y)^2 - x^2 + 2y^2$$

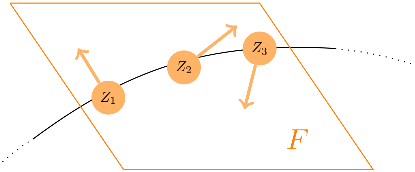
we produce an even better GAD of F:

$$x^2(2z_1 - 7z_2 - 5z_3) + y^2(4z_1 - 3z_2 - 2z_3) + (x+y)^2(z_1 + 5z_2 + 5z_3).$$

The scheme $Z' \subsetneq Z$ evincing the latter GAD is much shorter, and its Hilbert series is $[1, 5, 6, 6, \dots]$.

Perazzo

$$F \sim x^2 z_1 + y^2 z_2 + (x+y)^2 z_3.$$



(Negatively) Answers the question: Is the principal $(n+1)$-minor in the Hankel matrix of a concise form always invertible?

Figure: Tony, according to ChatGPT

U. Perazzo, *Sulle varietá cubiche la cui hessiana svanisce identicamente*, Giornale di Matematiche (Battaglini) 38, 1900, pp. 337–354.

Congratulations, Tony!

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