Commuting Jordan types

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Outline

Introduction

- We denote by $\mathcal{P}$ the set of all partitions of natural numbers and by $\mathcal{P}(n)$ the set of all partitions of natural number n.
- We first collect known facts about pairs of partitions $(P, Q) \in \mathcal{P}^2$ that are Jordan types of two commuting nilpotent matrices. We use notation $P \sim_{\mathcal{C}} Q$ for such pairs of partitions.
- Relation $\sim_{\mathcal{C}}$ is symmetric and reflexive, but it is not transitive.
- We write $P \succeq Q$ for the **dominance order** (also called the Bruhat order) on partitions. For partitions $P = (p_1, p_2, \ldots)$ and $Q = (q_1, q_2, \ldots)$ we have $P \succeq Q$ if and only if

$$\sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} q_i$$

for all k.

Universally Commuting Jordan Types

Definition

A partition $P \in \mathcal{P}(n)$ is called universally commuting *if $P \sim_{\mathcal{C}} Q$ for any partition $Q \in \mathcal{P}(n)$.*

Oblak 2012 characterised all universally commuting partitions. (See also Britnell, Wildon 2011 for an independent proof.)

Theorem

For $n \leq 3$ all partitions are universally commuting, while for $n \geq 4$ only partitions $(2^k, 1^l)$ are universally commuting.

Refinements

Definition

If P and Q are two partitions of n then P is a refinement *of Q if each part of Q is a sum of a subpartition of P and the subpartitions involved cover exactly all of the parts of P. A partition P is* almost-rectangular *if $P = (p^k, (p+1)^l)$ for some $p, k \geq 1$ and $l \geq 0$. We say that P is an* almost rectangular refinement *of Q if all subpartitions involved in the refinement are almost rectangular.*

Britnell and Wildon 2011 proved the following result.

Theorem

If Q and R are almost rectangular refinements of a partition P then $Q \sim_{\mathcal{C}} R$.

Other Known Facts

- Baranovsky 2001 proved for each partition P we have $P \sim_{\mathcal{C}} P^T$, where P^T is the conjugate partition of P.
- The only partitions that commute with partition (n) are almost rectangular partitions of n.
- Oblak 2012 proved:
  - Two distinct partitions of n that each have exactly two parts commute if and only if n is even, say $n = 2k$, and one of partitions is equal to (k, k) and the other is equal to $(k+1, k-1)$.
  - If $Q = (q_1, q_2, \ldots)$ is a partition commuting with partition (n, n) then either $Q = (2n)$ or $q_1 \leq n + 1$.
- Bogdanić, Djurić, Koljančić, Oblak, and Šivic 2024 [BDKOŠ] described all the partitions that commute with partition (n, n).

Partitions commuting with (n, n) - from [BDKOŠ]

	λ	*constraints*
(P1)	$[2n]^s$	$1 \leq s \leq 2n.$
(P2)	$([n+m]^{m+z}, [n-m]^{l-m+z})$	$0 \leq z \leq n-1,\ 0 < 2m \leq l < n-z,\ \lceil \frac{n+m}{m+z} \rceil \geq \lceil \frac{n-m}{l-m+z} \rceil + 1$ *or* $0 \leq z \leq n-1,\ 0 < m < n-z,\ l = n-z$
(P3)	$([n+\alpha']^{m+z}, [n-\alpha']^{l-m+z})$	$0 \leq z \leq n-1,\ 0 < m \leq l < n-z,\ l < 2m,\ n+l-m > \alpha(l+2z),\ \alpha' = l-m+(2m-l)\alpha$
(P4)	$([n+\alpha'']^{l-m+z}), [n-\alpha'']^{m+z})$	$0 \leq z \leq n-1,\ 0 < m \leq l < n-z,\ l < 2m,\ n+m+2z \leq \alpha(l+2z) < n+l+z,\ \alpha'' = m-(2m-l)\alpha$
(P5)	$([n]^{z+1}, [n]^{m+z})$	$0 \leq z \leq n-1,\ 2 \leq m \leq n-z$
(P6)	$((2\alpha)^{l-\beta}, (2\alpha-1)^{2z}, (2\alpha-2)^{\beta})$	$0 \leq z \leq n-1,\ 0 < m \leq l < n-z,\ l < 2m,\ \max\{l-m-z, 0\} \leq \beta < \min\{m+z, l\}$ *or* $0 \leq z \leq n-1,\ 0 \leq \beta < l < m \leq n-z$
(P7)	$((2\alpha-1)^{2l-m+z-\beta}, (2\alpha-2)^{2m-l+2z}, (2\alpha-3)^{\beta-m-z})$	$0 \leq z \leq n-1,\ 0 < m \leq l < n-z,\ l < 2m,\ \max\{m+z, l\} \leq \beta < \min\{2l-m+z, l+2z\}$
(P8)	$([n+\gamma']^{m+z-t}, [n-\gamma']^{m+z+t})$	$0 \leq z \leq n-1,\ 0 < m < \frac{n-z}{2},\ \lceil \frac{n+m+2z}{m+z} \rceil = \lceil \frac{n+m}{m+z} \rceil \geq 5,\ 1 \leq t \leq \min\{m, m-n+\lceil \frac{n-m}{m+z} \rceil (m+z)\},\ \gamma' = m - \lceil \frac{n-z}{m+z} \rceil t$
(P9)	$([n+\gamma'']^{m+z+t}, [n-\gamma'']^{m+z-t})$	$0 \leq z \leq n-1,\ 0 < m < \frac{n-z}{2},\ \lceil \frac{n-m}{m+z} \rceil \leq \lceil \frac{n-z}{m+z} \rceil,\ \lceil \frac{n+m}{m+z} \rceil \geq 4,\ t \geq 1,\ n+z+t \leq \lceil \frac{n+m}{m+z} \rceil (m+z) \leq n+2m+z-t,\ \gamma'' = m + \lceil \frac{n-m-2z}{m+z} \rceil t$

TABLE 1. A complete list of the nilpotent orbits $\mathcal{O}_\lambda$ having a non-empty intersection with $\mathcal{N}(B)$, $B \in \mathcal{O}_{(n,n)}$. Here, we use the notation $\alpha = \lceil \frac{n+z}{l+2z} \rceil$ and $\beta = \alpha(l+2z) - (n+z)$.

Setup

- Suppose B is a nilpotent matrix and $P \in \mathcal{P}$ is its Jordan type.
- The set $\mathcal{N}_B$ of all nilpotent matrices commuting with B is an irreducible algebraic variety (Basili, 2003).
- Its intersection with one of the nilpotent orbits is a dense open subset in the Zariski topology. The Jordan type of the orbit is denoted by $\mathcal{D}(P)$.
- For each partition P we obviously have $P \sim_{\mathcal{C}} \mathcal{D}(P)$.
- The image of the map $\mathcal{D} : \mathcal{P} \to \mathcal{P}$ on the set of all partitions of natural numbers $\mathcal{P}$ is equal to the set $\mathcal{Q}$ of all super-distinct, also called Rogers-Ramanujan, partitions, i.e., partitions whose parts differ by at least 2 (Basili, Iarrobino, 2008).
- The map $\mathcal{D}$ is an idempotent (K., Oblak, 2009).

Dominance Order and $\mathcal{D}(P)$

The following is a known fact, stated e.g. by Khatami 2024.

Proposition ($\mathcal{D}(P)$-maximal)

Suppose that P and Q are partitions of n. If $P \sim_{\mathcal{C}} Q$ then $\mathcal{D}(P) \succeq Q$.

Sketch of Proof:

- The intersection of $\mathcal{N}_B$ with the nilpotent orbit $\mathcal{O}_{\mathcal{D}(P)}$ is a dense open subset of $\mathcal{N}_B$.
- Since $\mathcal{N}_B$ is irreducible we have

$$\mathcal{N}_B = \overline{\mathcal{O}_{\mathcal{D}(P)} \cap \mathcal{N}_B} = \overline{\mathcal{O}_{\mathcal{D}(P)}} \cap \mathcal{N}_B.$$

- This implies that if $\mathcal{O}_Q$ is any orbit such that $\mathcal{N}_B \cap \mathcal{O}_Q \neq \emptyset$, then $\mathcal{O}_Q \subset \overline{\mathcal{O}_{\mathcal{D}(P)}}$.
- By the Gerstenhaber-Hesselink Theorem we have $\mathcal{D}(P) \succeq Q$.

Corollary

For each partition $Q \in \mathcal{Q}$ we have $Q \succeq P$ for any $P \in \mathcal{D}^{-1}(Q)$.

The Box Conjecture/Theorem

- Iarrobino, Khatami, Van Steirteghem, and Zhao 2018 conjectured that for each $Q \in \mathcal{Q}$ the inverse image $\mathcal{D}^{-1}(Q)$ is a 'box'.
- Last year, Irving, K., Mastnak, [IKM] proved the Conjecture using the Burge bijection (of 1981) between the set of all partitions and the set of all words in two letters α and β that end in a single α.

Dominance order and the part with the maximal number of parts in $\mathcal{D}^{-1}(Q)$

For a partition $Q \in \mathcal{Q}$ there is a unique partition with the maximal number of parts in the box $\mathcal{D}^{-1}(Q)$. We denote this unique partition by $Q_{\min}$.

Note

Example 3 of [IKM] shows that in general the partition $Q_{\min}$ is not the minimal partition in the box $\mathcal{D}^{-1}(Q)$ in the dominance order. So, $Q_{\min}$ is not the minimal element of the box $\mathcal{D}^{-1}(Q)$ and a dual of Corollary does not hold.

$Q_{\min}$ is not minimal in $\mathcal{D}^{-1}(Q)$ - from [IKM]

(i_1, i_2, i_3)	code ω	partition $\Omega^{-1}(\omega)$	# parts
$(1,1,1)$	$\alpha\alpha\beta\alpha\alpha\alpha\beta\alpha\alpha\beta\alpha$	$[10,7,3]$	3
$(2,1,1)$	$\alpha\beta\beta\alpha\alpha\alpha\beta\alpha\alpha\beta\alpha$	$[10,7,2,1]$	4
$(3,1,1)$	$\beta\beta\beta\alpha\alpha\alpha\beta\alpha\alpha\beta\alpha$	$[10,7,1^3]$	5
$(1,2,1)$	$\alpha\alpha\beta\alpha\alpha\beta\beta\alpha\alpha\beta\alpha$	$[10,5,3,2]$	4
$(2,2,1)$	$\alpha\beta\beta\alpha\alpha\beta\beta\alpha\alpha\beta\alpha$	$[10,4,3,2,1]$	5
$(3,2,1)$	$\beta\beta\beta\alpha\alpha\beta\beta\alpha\alpha\beta\alpha$	$[10,4,3,1^3]$	6
$(1,3,1)$	$\alpha\alpha\beta\alpha\beta\beta\beta\alpha\alpha\beta\alpha$	$[10,5,2^2,1]$	5
$(2,3,1)$	$\alpha\beta\beta\alpha\beta\beta\beta\alpha\alpha\beta\alpha$	$[10,5,2,1^3]$	6
$(3,3,1)$	$\beta\beta\beta\alpha\beta\beta\beta\alpha\alpha\beta\alpha$	$[10,5,1^5]$	7
$(1,1,2)$	$\alpha\alpha\beta\alpha\alpha\alpha\beta\alpha\beta\beta\alpha$	$[9,5,3^2]$	4
$(2,1,2)$	$\alpha\beta\beta\alpha\alpha\alpha\beta\alpha\beta\beta\alpha$	$[9,4^2,2,1]$	5
$(3,1,2)$	$\beta\beta\beta\alpha\alpha\alpha\beta\alpha\beta\beta\alpha$	$[9,4^2,1^3]$	6
$(1,2,2)$	$\alpha\alpha\beta\alpha\alpha\beta\beta\alpha\beta\beta\alpha$	$[9,5,2^3]$	5
$(2,2,2)$	$\alpha\beta\beta\alpha\alpha\beta\beta\alpha\beta\beta\alpha$	$[9,4,3,2,1^2]$	6
$(3,2,2)$	$\beta\beta\beta\alpha\alpha\beta\beta\alpha\beta\beta\alpha$	$[9,4,3,1^4]$	7
$(1,3,2)$	$\alpha\alpha\beta\alpha\beta\beta\beta\alpha\beta\beta\alpha$	$[9,5,2^2,1^2]$	6
$(2,3,2)$	$\alpha\beta\beta\alpha\beta\beta\beta\alpha\beta\beta\alpha$	$[9,5,2,1^4]$	7
$(3,3,2)$	$\beta\beta\beta\alpha\beta\beta\beta\alpha\beta\beta\alpha$	$[9,5,1^6]$	8

FIGURE 1. The elements of $\mathfrak{D}^{-1}(10,7,3)$ and the corresponding Burge codes, indexed as in Corollary 2 by their coordinates $(i_1,i_2,i_3) \in [1,3]\times[1,3]\times[1,2]$.

Negative Results

When Partitions Do Not Commute

Among the negative results, i.e., which pairs of partitions are not pairs of Jordan types of two commuting matrices, we state first an easy consequence of Proposition.

Theorem

If Q and R are two distinct partitions of n in $\mathcal{Q}$, then $Q \not\sim_{\mathcal{C}} R$.

Proof.

- Since $\mathcal{D}$ is idempotent and $Q, R \in \operatorname{Im} \mathcal{D}$, we have $Q = \mathcal{D}(Q)$ and $R = \mathcal{D}(R)$.
- Assume that the nilpotent orbit $\mathcal{O}_Q$ intersects $\mathcal{N}_B$, where R is the Jordan type of B. We want to obtain a contradiction.
- By Proposition our assumption implies that $R = \mathcal{D}(R) \succeq Q$.
- By reversing the roles of R and Q we also obtain that $Q = \mathcal{D}(Q) \succeq R$, which is possible if and only if $Q = R$.

Open questions

- Determine all commuting Jordan types.
- The Burge correspondence works nicely on $\mathcal{P}$. Since the Burge generators do not respect the length of P, it seems less intuitive on $\mathcal{P}(n)$ for partitions in distinct 'boxes'. Recall that $\mathcal{P}(n)$ is the disjoint union of all the 'boxes' parametrized by R-R partitions of n. One can formulate two sub-questions:
  - When do two partitions from the same 'box' commute?
  - When do two partitions from distinct 'boxes' commute?
- Is there an analoguous 'Burge correspondence' for other simple Lie algebras (B, C, D, . . . types)?

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