Solving Linear Equations

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We introduce the notion of an "initial condition" for a module M over a commutative Noetherian local ring $(A, \mathfrak{m})$, allowing for a recursive construction of its "solution modules". If M has zero-dimensional support, such as the residue field of A, we demonstrate that the solution module $E(M)$ is its "linear closure", turning out to be an injective hull of M. The construction of $E(M)$ for finitely generated M, hence injective hull of M, is explicit and computable, devoid of the need for Zorn's lemma. As an application, we improve Baer's criterion for a module N with zero-dimensional support to be injective: If any A-homomorphism from $\mathfrak{m}$ to N lifts to A, then N is injective. For the case that A is Artinian and M is its residue field, we demonstrate how systems of linear equations are explicitly solved to obtain an injective hull.